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MARKOV NEIGHBORHOODS FOR ZERO-DIMENSIONAL BASIC SETS

BY

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ABSTRACT. We extend the local stable and unstable laminations for a zero-dimensional basic set to semi-invariant laminations of a neighborhood, and use these extensions to construct the appropriate analog of a Markov partition, which we call a Markov neighborhood. The main applications we give are in the perturbation theory for stable and unstable manifolds; in particular, we prove a transversality theorem. For these applications we require not only that the basic sets be zero dimensional but that they satisfy certain tameness assumptions. This leads to global results on improving stability properties via small isotopies.

Introduction. The earliest substantial result on structural stability of diffeomorphisms was due to Palis and Smale **[PS]**. They proved structural stability of Morse-Smale diffeomorphisms f by carefully constructing charts about the periodic orbits in which f has a particularly nice form, so that the charts overlap in a compatible fashion. A conjugacy between f and a perturbation g is then easily expressed using the charts for f and g. These charts are constructed in terms of invariant tubular families: local laminations invariant under f which extend the stable and unstable manifolds. (Laminations are, roughly, C^0 foliations with continuous tangent bundles; see §1. The standard definitions and results of differentiable dynamics are in **[S]** or **[B]**, and are reviewed in §§ 3 and 7.)

This approach to structural stability was extended by de Melo **[M]** to AS (Axiom A, strong transversality) diffeomorphisms on surfaces, but was abandoned by Robbin **[Ro]** and Robinson **[R]** in their definitive treatments. In fact, it is unknown whether such tubular families can be constructed for arbitrary basic sets.

Another important ingredient in stability theory is the notion of canonical coordinates. This was introduced by Smale for the Ω-stability theorem **[S, S1]** and again refers to a system of charts, this time on the nonwandering set, induced by the local invariant laminations. This was refined by Bowen into a powerful tool for studying basic sets: Markov partitions. An elegant presentation, including the Ω-stability theorem, is in **[B]**.

It is our intention here to merge and extend these techniques to study neighborhoods of zero-dimensional basic sets. Thus we construct invariant tubular families

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for such basic sets and, using these, we extend a Markov partition to a neighborhood. (Newhouse **[N2]** gave a similar construction, but only in two dimensions.) We call these neighborhoods Markov neighborhoods. They give a standardized, simple form for diffeomorphisms near zero-dimensional basic sets, and in a sequel to this paper we shall construct them compatibly for AS diffeomorphisms with $\dim \Omega = 0$, thus recapturing the original Palis-Smale proof of structural stability.

The class of AS diffeomorphisms with $\dim \Omega = 0$ is particularly important in light of the results of Shub and Sullivan **[SS]**, establishing C^0 density of such diffeomorphisms in every isotopy class. As an application of the Markov neighborhood theorem we consider problems of smoothing invariant laminations and of putting them in general position, and find that tameness assumptions are necessary before we can make any progress. This tameness refers not to the embedding of the basic set in M, but to its embedding in the invariant laminations $W^{\pm}$. Assuming such tameness we can construct isotopies to smooth the invariant laminations and make the laminations for different basic sets transverse, without changing Ω-stability type. As a result we obtain (§7) a weak version of the Kupka-Smale theorem and a relative version of the Shub-Sullivan isotopy theorem which does not require fitted handle decompositions. In the sequel promised above we shall show that the AS diffeomorphisms with $\dim \Omega = 0$ and suitable tameness properties are exactly the fitted diffeomorphisms produced in **[SS]**.

Outline. §1. A presentation of disk families and laminations. Much of this is known in some version, but a systematic development is not available elsewhere.

§2. Construction of tubular families transverse to laminations which are transversely zero dimensional; that is, which are locally of the form $A \times D^k$ with $\dim A = 0$.

§3. Semi-invariant tubular families in a neighborhood of a zero-dimensional basic set, reinterpreting **[R]** in view of §2.

§4. Construction of Markov neighborhoods.

§5. Various notions of tameness; proof that zero-dimensional basic sets are tame in M; topological conjugacy for basic sets with suitable tameness in $W^{\pm}$.

§6. Perturbations to achieve smoothness and transversality of invariant laminations.

§7. Global perturbations: the weak Kupka-Smale theorem, and the relative Shub-Sullivan theorem.

§8. Examples illustrating the need for tameness and for C^1-large perturbations.

1. Disk families and laminations. Our purpose here is to develop systematically some notions necessary for stable manifold and stability theory. Much of this is folklore or is scattered through the literature. In particular, similar material can be found in **[HPPS, HPS, R]**.

Most of our terminology follows **[H]**. *Smooth* always means C^1. Unless the contrary is stated, all spaces are Hausdorff and all maps are continuous. For the entire paper M shall denote a boundaryless paracompact separable C^∞ n-manifold equipped with a C^∞ Riemannian metric, which we use to norm all bundles derived from TM. If π: $E \to A$ is a normed vector bundle and ρ: $A \to \mathbf{R}^+$ we set $E(\rho) = \{\xi \in E: \|\xi\| \leq \rho(\pi\xi)\}$. Note that if A is paracompact then the sets $E(\rho)$, as ρ varies, form a neighborhood basis for the zero section.

Disk families. We first define a class of bundle maps. Suppose $E, F \to A$ are vector bundles, $U \subset E$ is open, ϕ: $U \to F$ covers the identity, and the restrictions ϕ_a: $U_a = U \cap E_a \to F_a$ are smooth for all $a \in A$. For $\xi \in U_a$ we have the *fiber derivative* $FD\phi(\xi)$ defined as $D\phi_a(\xi)$: $E_a \to F_a$, and we say ϕ is FC^1 (*fiberwise smooth*) iff $FD\phi$ is continuous as a map from U to the linear map bundle $L(E, F)$. If U is not open we say ϕ is FC^1 iff it extends to an FC^1 map on a neighborhood of U. An FC^1 map ϕ is an FC^1 *embedding* iff it is a topological embedding and each ϕ_a is a smooth embedding; ϕ is an FC^1 *isomorphism* of U onto ϕU iff it has an FC^1 inverse. The following is a useful formulation of the inverse function theorem; in it $FD_0\phi$ denotes the bundle map given by $FD\phi(0_a)$ on each fiber.

(1.1) THEOREM. *Suppose E_0, $T \to A$ are normed vector bundles, A is paracompact, U_0 is a neighborhood of the zero section in E_0, and ϕ_0: $U_0 \to T$ is an FC^1 embedding preserving the zero section. Suppose E, F are subbundles of T and $T = E \oplus F = \operatorname{Im}(FD_0\phi_0) \oplus F$, and let P: $E \oplus F \to E$ be the projection. Then there are ρ: $A \to \mathbf{R}^+$, an FC^1 map w: $E(\rho) \to F$, an FC^1 embedding ϕ: $E(\rho) \to T$, and an FC^1 isomorphism Φ: $P^{-1}E(\rho) \to P^{-1}E(\rho)$, with $\phi(\xi) = \xi + w(\xi)$, $\Phi(\zeta) = \zeta + wP\zeta$, such that $\operatorname{Im}\phi$ is a neighborhood of the zero section in $\operatorname{Im}\phi_0$.*

PROOF. Note that $L = P \circ FD_0\phi_0$ is a linear isomorphism of E_0 onto E. Setting $\psi = P\phi_0 L^{-1}$ we have $FD_0\psi = 1$, so, restricting ψ to a small enough $E(\rho_0) \subset LU_0$, we have $\|FD\psi - 1\| \leq \frac{1}{2}$. The inverse function theorem (see [**HP**, 1.5, 1.6]) implies that $\psi E(\rho_0) \supset E(\rho)$ with $\rho = \frac{1}{2}\rho_0$, and that ψ has an inverse which is smooth on each fiber. Using the fiber contraction theorem [**HP**, 6.1a] instead of the Banach contraction principle in [**HP**, 1.5] shows that ψ^{-1} is continuous so, by the formula $FD\psi^{-1}(\xi) = [FD\psi(\psi^{-1}\xi)]^{-1}$, it is FC^1. Hence $P\phi_0 | L^{-1}E(\rho_0)$ has an FC^1 inverse $L^{-1}\psi^{-1}$ whose domain contains $E(\rho)$. Define $w(\xi) = \phi_0 L^{-1}\psi^{-1}\xi - \xi$; then $Pw = 0$ so w maps $E(\rho)$ into F. Define ϕ, Φ as in the statement of the theorem. The only thing to check is that Φ has an FC^1 inverse, which is clear from $\Phi^{-1}(\zeta) = \zeta - wP\zeta$. □

Now recall that $A \subset M$ is *locally closed* iff $A = F \cap G$ with F closed and G open. For subsets of M this condition is equivalent to local compactness and it implies paracompactness. Also recall that there is a C^∞ diffeomorphism Exp from a neighborhood of the zero section in TM onto a neighborhood of the diagonal in $M \times M$ given by $\xi \mapsto (\pi\xi, \exp\xi)$ where exp is the usual exponential map. Suppose $A \subset M$ is locally closed and X is a function assigning a smooth k-disk $X(a)$ in M to each $a \in A$. We say such X is a *k-disk family* iff there is a splitting $E \oplus F$ of $TM|A$, a map ρ: $A \to \mathbf{R}^+$, and an FC^1 map w: $E(\rho) \to F$ so that $w(0) = 0$, the map ϕ: $\xi \mapsto \xi + w(\xi)$ carries $E(\rho)$ into the domain of Exp, and

$$\operatorname{Exp}(\operatorname{Im}\phi) = \bigcup \{a \times X(a) : a \in A\}.$$

Such w, or the corresponding ϕ, is called a *parameterization* of X.

Usually we shall be content to *shrink* X, that is, to replace X by a k-disk family X_0 through A with $X_0(a) \subset X(a)$ for all $a \in A$. Then (1.1) allows considerable freedom in parametrizing shrinkings of X. Moreover, much more general notions of disk families shrink, via (1.1), to our definition.

For a disk family X through A we write $T_0X = \bigcup\{T_aX(a)\colon a \in A\}$. If ϕ: $E(\rho) \to TM|A$ parametrizes X then T_0X is the image of $FD_0\phi$, and hence is a subbundle of $TM|A$. We now collect some basic facts about disk families.

(1.2) LEMMA. *A disk family through A extends to a neighborhood of A.*

PROOF. Take a parametrization w: $E(\rho) \to F$. The splitting $E \oplus F$, and the map ρ: $A \to \mathbf{R}^+$, extend to a neighborhood of A. (See **[HPPS**, 4.4**]** and remember that A is a closed subset of some open neighborhood.) In a bundle chart w can be regarded as a map into the Banach space of C^1 maps $D^k \to \mathbf{R}^{n-k}$, and this extends to a neighborhood by the Dugundji extension theorem **[D]**. Now patch together these local extensions with a partition of unity. $\square$

An *isotopy* of disk families is a collection $\{X_t\colon t \in I\}$ of disk families through the same set A such that $X(a, t) = X_t(a) \times t$ defines a disk family through $A \times I$ in $M \times \mathbf{R}$. The following is analogous to uniqueness of tubular neighborhoods.

(1.3) THEOREM. *Suppose X_0, X_1 are disk families through A and $T_0X_0 \oplus F = T_0X_1 \oplus F = TM|A$ for some F. Then there is an isotopy X_t from X_0 to X_1 with $T_0X_t \oplus F = TM|A$ for all t, and $X_t(a) \subset X_1(a)$ for all t whenever $X_0(a) \subset X_1(a)$. Moreover, given a disk family Z through A there is such an X_t which also satisfies $X_t(a) \subset Z(a)$ for all t whenever $X_0(a) \cup X_1(a) \subset Z(a)$.*

PROOF. Parametrize X_i by ϕ_i: $E_i(\rho_i) \to E_i \oplus F_i$. If $E_0 = E_1$, $F_0 = F_1 = F$ then take $\rho \leqslant \min(\rho_0, \rho_1)$, perform preliminary isotopies to shrink the domain of ϕ_i to $E_i(\rho)$, and then use $\phi_t = (1 - t)\phi_0 + t\phi_1$.

Next suppose $F_0 = F_1 = F$ but $E_0 \neq E_1$. We reduce this to the preceding case by isotoping X_0 to a disk family parametrized by $E_1(\rho_1^*) \to F$. Let P: $E_0 \oplus F \to E_0$ be the projection. Since $E_1 \oplus F = E_0 \oplus F$ there is a linear map G: $E_0 \to F$ so that, for $t \in [0, 1]$, $L_t = 1 + tGP$ is a linear automorphism of $TM|A$ with $L_0 = 1$, $L_1E_0 = E_1$. Set $E_t = L_tE_0$ and note that $E_t \oplus F = TM|A$ for all t, since $L_tF = F$. In $TM|A \times T\mathbf{R}|I$ set $E_0^* = E \times 0$, $F^* = F \times T\mathbf{R}|I$, $E^* = \bigcup\{E_t \times 0_t\colon t \in I\}$, and define $\phi_0^*(\xi, 0) = (\phi_0\xi, 0)$ for $\xi \in E_0(\rho)$. Then (1.1) provides $\phi^* = 1 + w^*$: $E^*(\rho^*) \to E^* \oplus F^*$ with $\operatorname{Im}\phi^* \subset \operatorname{Im}\phi_0^*$, so $\phi^*(\xi, 0_t) = (\phi_t^*\xi, 0_t)$. Write $\rho^*(a, t) = \rho_t^*(a)$. Then, as above, we can isotope ϕ_0 to $\phi_0|E_0(\rho_0^*)$, and then ϕ_t^* defines an isotopy to $\phi_1^*|E_1(\rho_1^*)$. Note that in this entire process we isotope each $X_0(a)$ in itself.

Finally we reduce the general case to the preceding, by isotoping X_0 to a disk family given by $\tilde{E}_1(\tilde{\rho}_1) \to F$, and then treating X_1 similarly. Using $T_0X_0 \oplus F = T_0X_0 \oplus F_0$ we get a family L_t of automorphisms of $TM|A$ with $L_0 = 1$, $L_1F_0 = F$, and $L_tF_0 \oplus L_tE_0 = L_tF_0 \oplus T_0X_0 = TM|A$ for all t. Arguing as above, we find the isotopy in the form $\tilde{E}_t(\tilde{\rho}_t) \to L_tF_0$ with $\tilde{E}_t = L_tE_0$.

If Z was specified take E_Z and an FC^1 isomorphism Φ_Z as in (1.1). Now by (1.1) there are disk families $\tilde{X}_i$ parameterized by $\tilde{\phi}_i$: $\tilde{E}_i(\tilde{\rho}_i) \to TM|A$ with $\operatorname{Im}\tilde{\phi}_i \subset \operatorname{Im}\Phi_Z^{-1}\phi_i$, and $T_0\tilde{X}_i \oplus FD_0\Phi_Z^{-1}F = TM|A$. Hence we can apply the argument above to find an isotopy $\tilde{X}_t$ from $\tilde{X}_0$ to $\tilde{X}_1$ parameterized by $\tilde{\phi}_t$: $\tilde{E}_t(\tilde{\rho}_t) \to TM|A$. From the construction of $\tilde{X}_t$ above we have

$$X_0(a) \cup X_1(a) \subset Z(a) \Rightarrow \operatorname{Im}\tilde{\phi}_{0a} \cup \operatorname{Im}\tilde{\phi}_{1a} \subset E_Z$$
$$\Rightarrow \operatorname{Im}\tilde{\phi}_{ta} \subset E_Z \Rightarrow \operatorname{Im}(\exp \Phi_Z\phi_{ta}) \subset Z(a).$$

By (1.1) there is a disk family X'_t with $\exp_a^{-1}X'_t(a) \subset \operatorname{Im}(\Phi_Z\tilde{\phi}_{ta})$, so $X'_i(a) \subset X_i(a)$ for $i = 0, 1$. But we have already shown that X_i is isotopic to X'_i, moving each $X_i(a)$ inside itself. $\square$

(1.4) Corollary. *For $i = 1, 2$ suppose X_i is a disk family through an open A_i, and suppose $X_1(a) \cap X_2(a)$ is a neighborhood of a in $X_i(a)$ for all $a \in A_0 = A_1 \cap A_2$. Let B be a neighborhood of* $\operatorname{Cl} A_0$. *Then there is a disk family X through $A = A_1 \cup A_2$ with $X(a) = X_i(a)$ for $a \in A_i \setminus \dot{B}$, $X(a) \subset X_i(a)$ for $a \in A_i \setminus A_0$, and $X(a) \subset X_1(a) \cup X_2(a)$ for $a \in A_0$.*

Proof. $E = T_0X_1 \cup T_0X_2$ is a subbundle of $TM|A$. By (1.3) we can isotope each X_i inside itself to a disk family parameterized as ϕ_i: $E|A_i(\rho_i) \to TM|A_i$, and using this isotopy and a bump function vanishing off B we can construct $\tilde{X}_i$ given by X_i on $A_i \setminus B$ and parameterized by ϕ_i on a neighborhood of $\operatorname{Cl} A_0 \cap A_i$ in A_i. Then we find ρ: $A \to \mathbf{R}^+$ with $\rho \leqslant \max(\rho_1, \rho_2)$ on A_0, $\rho \leqslant \rho_i$ on $A_i \setminus A_0$, and obtain X from $\tilde{X}$ by using a parameterization ϕ: $E(\rho) \to TM$ near A_0, where $\phi_a = \phi_i|E(\rho)_a$ if $\rho(a) \leqslant \rho_i(a)$. $\square$

(1.5) Lemma. *If X, Y are disk families through A and $X(a) \pitchfork_a Y(a)$ for all $a \in A$ then there is a shrinking Y_1 of Y and a disk family Z through A with $X(a) \cap Y_1(a) \subset Z(a) \subset X(a) \cap Y(a)$ for all $a \in A$.*

Proof. Parameterize X as w_X: $E_X(\rho_X) \to F_X$ and define Φ_X as in (1.1). Parameterize Y as ϕ_Y: $E_Y(\rho_Y) \to TM|A$ and shrink ρ_Y so that $\psi = \Phi_X^{-1}\phi_Y$ is well defined. Set $E = \operatorname{Im} FD_0\psi$. Since $FD_0\Phi_X^{-1}(T_0X + T_0Y) = E_X + E = TM|A$ we can find $F \subset E_X$ with $E \oplus F = TM|A$; for example, $E_X \cap (E_X \cap G)^{\perp}$ will work. So we can apply (1.1) to find ϕ: $E(\rho) \to E \oplus F$ with $\operatorname{Im}\phi \subset \operatorname{Im}\psi$. Let $\phi_0 = \phi|(E \cap E_X)(\rho)$. Since $F \subset E_X$ we have $\operatorname{Im}\phi_0 \subset E_X \cap \operatorname{Im}\psi$, so $\operatorname{Im}\Phi_X\phi_{0a} \subset \exp_a^{-1}(X(a) \cap Y(a))$. Define Z by applying (1.1) to $\Phi_X\phi_0$. Then there is a neighborhood U of the zero section with $\exp_a^{-1}Z(a) = U_a \cap \operatorname{Im}\Phi_X\phi_0$. Take Y_1 so that $\exp_a^{-1}Y_1(a) \subset U_a$. $\square$

We need a version of (1.5) which considers intersections $Y(a) \cap X(b)$ for $a \neq b$. A continuous map $(a, b) \mapsto [a, b] \in M$ defined on a neighborhood V of the diagonal in $A \times A$ is called an (X, Y) *canonical map* iff $Y(a) \pitchfork X(b)$ and $Y(a) \cap X(b) = [a, b]$ for all $(a, b) \in V$.

(1.6) Theorem. *Suppose X, Y are disk families through A with $T_0X \oplus T_0Y = TM|A$. Then for some shrinking X_1 of X there is an (X_1, Y) canonical map.*

Proof. Let p_i: $A \times A \to A$ be projection on the ith coordinate. If $E \to A$ is a bundle we write E_i for the pullbacks p_i^*E; these can be identified as $E_1 = E \times A$,

$E_2 = A \times E$. We write $T_i = p_i^* TM|A$. If λ: $A \to B$ we write $\lambda_i = \lambda \circ p_i$, and if ϕ: $E \to F$ is a bundle map we write ϕ_i for the pullback $E_i \to F_i$. In the proof we replace $A \times A$ with successively smaller neighborhoods V of the diagonal, and we still write E_i instead of $E_i | V$, etc.

For V small enough we can use Exp to construct an FC^1 isomorphism H: $U_1 \to U_2$ between neighborhoods of the zero sections in T_i so that $H(\xi, b) = (a, \eta)$ if and only if $\exp_a \xi = \exp_b \eta$. Parameterize Y as $E(\rho) \to F$ and take Φ as in (1.1); parameterize X by ψ: $G(\sigma) \to TM|A$. If V is small enough, H carries the zero section into the interior of dom Φ_2^{-1}, so, after shrinking σ, $g_0 = \Phi_2^{-1} H \psi_1$: $G_1(\sigma_1) \to T_2$ is well defined, as is $g = g_0 - g_0(0)$. Over the diagonal Im $FD_0 g \oplus E_2 = T_2$ so this remains true for V small enough. Thus we can apply (1.1) to find f: $F_2(\alpha) \to T_2$ with Im $f \subset$ Im g and $P_2 f = 1$ where P: $E \oplus F \to F$ is the projection. Select $\sigma^1 \leqslant \sigma_1$ so that $g(G_1(\sigma^1)) \subset$ Im $f \cap T_2(\frac{1}{2}\rho_2)$. Replacing V by $\{(a, b)$: $\sigma^1(a, b) \geqslant \frac{1}{2}\sigma^1(a, a)\}$ and defining $\tau(a) = \frac{1}{2}\sigma^1(a, a)$, we have $\tau_1 \leqslant \sigma^1$. We set $\tilde{g}_0 = g_0 | G_1(\tau_1)$, $\tilde{g} = g | G_1(\tau_1)$. Similarly we find β: $A \to R^+$ with $\beta_2 \leqslant \alpha$, so that $\tilde{f} = f | F_2(\beta_2)$ maps into Im $\tilde{g}$. Now shrink V once more, so that $g_0(0) \in P_2^{-1} F_2(\beta_2) \cap T_2(\frac{1}{2}\rho_2)$.

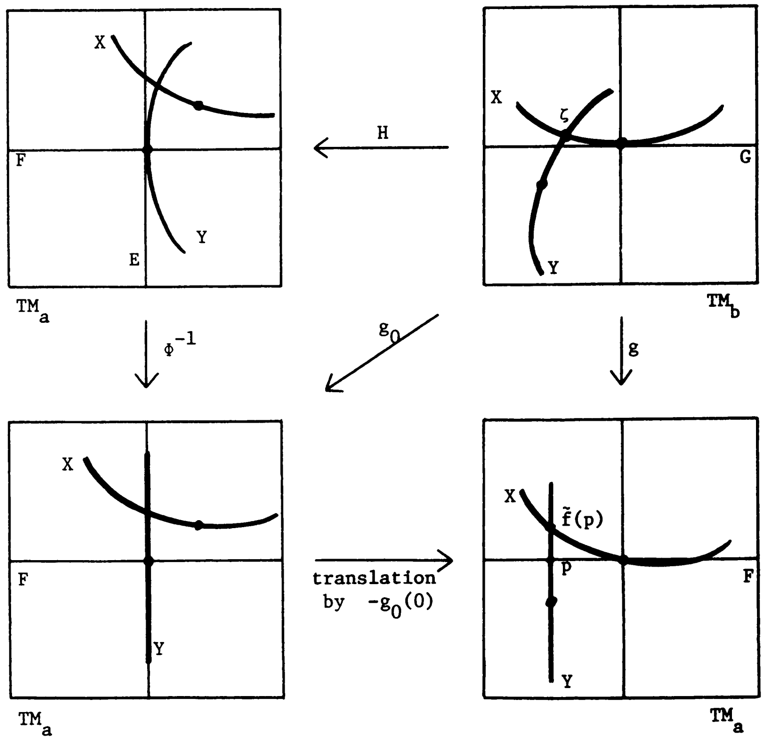


FIGURE 1.1.
Defining $[a, b]$. The disks X, Y (and their centers) are actually $\exp_b^{-1} X(b)$, $\exp_b^{-1} Y(a)$ in TM_b, and their images under H, g_0, g in TM_a. In the lower right $p = -P(g_0(0))$.

We define X_1 by $\psi | G(\tau)$. For $(a, b) \in V$ we have $z \in Y(a) \cap X_1(b)$ iff there are $\eta \in E(\rho)_a$, $\xi \in G(\tau)_b$ with $\exp_a \Phi(\eta) = \exp_b \psi(\xi)$, which holds iff $\tilde{g}_0(a, \xi) \in E_2(\rho_2)_{(a,b)}$. This is equivalent to $P_2 \tilde{g}_0(a, \xi) = 0$, since $\tilde{g}_0 = g_0(0) + \tilde{g}$ maps into $T_2(\rho_2)$. Also, $P_2 \tilde{g}_0(a, \xi) = 0$ iff $P_2 \tilde{g}(a, \xi) = -P_2 g_0(a, 0)$. Since $-P_2 g_0(a, 0) \in F_2(\beta_2)$ this is equivalent to $\tilde{g}(a, \xi) = \tilde{f}(-P_2 g_0(a, 0))$.

So we have shown $Y(a) \cap X_1(b)$ consists of the single point $\exp_b \zeta$, where $(a, \zeta) = \Phi_2[\tilde{f}(-P_2 g_0(a, 0_b)) + g_0(a, 0_b)]$. □

Laminations. The unqualified term submanifold always means embedded submanifold. By an *immersed submanifold* we mean a subset $\mathfrak{L}$ of M with a C^1 manifold structure (usually nonseparable) such that the inclusion $\mathfrak{L} \to M$ is an immersion. For $x \in \mathfrak{L}$ we write $\mathfrak{L}(x)$ for the intrinsic component of $\mathfrak{L}$ containing x and call it the leaf of $\mathfrak{L}$ through x. Henceforth we refer to the intrinsic structure of $\mathfrak{L}$ only by referring to its leaves (with the exception that the dimension of $\mathfrak{L}$ means the leaf dimension). Other references to the topology of $\mathfrak{L}$ refer to its status as a subset of M.

We say a k-dimensional immersed submanifold $\mathfrak{L}$ is a *lamination* iff there is a k-disk family X through $\mathfrak{L}$ with $X(x) \subset \mathfrak{L}(x)$ for all $x \in \mathfrak{L}$ (so, in particular, $\mathfrak{L}$ is locally closed in M). Such X is called a *plaquation* for $\mathfrak{L}$. If $\mathfrak{L}$ is compact this definition coincides with that in **[HPS, §7]**, but our primary interest is in noncompact laminations. Our results on disk families translate to facts about laminations. Thus $T\mathfrak{L} = \bigcup \{T\mathfrak{L}(x)\colon x \in \mathfrak{L}\}$ is a subbundle of $TM|\mathfrak{L}$. If $\mathfrak{L}$, $\mathfrak{N}$ are transverse laminations (meaning $\mathfrak{L}(x) \pitchfork \mathfrak{N}(x)$ for all x) then, by (1.5), $\mathfrak{L} \cap \mathfrak{N}$ has a lamination structure whose leaves are the components of $\mathfrak{L}(x) \cap \mathfrak{N}(x)$. Similarly, if $\mathfrak{L}$ and $\mathfrak{N}$ are open in M and *compatible* (meaning that $\mathfrak{L}(x) \cap \mathfrak{N}(x)$ is open in both $\mathfrak{L}(x)$ and $\mathfrak{N}(x)$ for all $x \in \mathfrak{L} \cap \mathfrak{N}$) then, by (1.4), $\mathfrak{L} \cup \mathfrak{N}$ has a lamination structure whose leaves are maximal connected unions of leaves of $\mathfrak{L}$ and $\mathfrak{N}$. The canonical map of (1.6) is promoted to *canonical coordinates*:

(1.7) THEOREM. *Suppose $\mathfrak{L}$, $\mathfrak{N}$ are transverse laminations of complementary dimensions, and set $A = \mathfrak{L} \cap \mathfrak{N}$. If X, Y are plaquations of $\mathfrak{L}, \mathfrak{N}$ for which the (X, Y) canonical map $[\,,]$ is defined then there are shrinkings X_0, Y_0 of $X|A$, $Y|A$ such that, for all $a \in A$, if $P_a = X_0(a) \cap \mathfrak{N}$, $Q_a = \mathfrak{L} \cap Y_0(a)$, then $[\,,]$ restricted to $P_a \times Q_a$ is a homeomorphism onto a neighborhood of a in A.*

PROOF. Let V_0 be the neighborhood of the diagonal in $A \times A$ on which $[\,,]$ is defined. Using the methods of (1.6) the reader may easily prove

(1.8) LEMMA. *If X is a plaquation for a lamination $\mathfrak{L}$ then there is a neighborhood V of the diagonal in $\mathfrak{L} \times \mathfrak{L}$ such that if $(a, b) \in V$ then $a \in X(b)$ if and only if $b \in X(a)$.*
□

Apply this to both X and Y to find such a neighborhood, V_1. By continuity of $[\,,]$ we can find a neighborhood V of the diagonal in $A \times A$ such that $(a, x), (a, y) \in V \Rightarrow (x, y), ([x, y], a), (a, [x, y]) \in V_0 \cap V_1$. Now choose X_0, Y_0 so that $P_a \times Q_a \subset V$ for all $a \in A$. Then for $(x, y) \in P_a \times Q_a$ the points $x' = [[x, y], a]$ and $y' = [a, [x, y]]$ are well defined, and using (1.8) we conclude $x = x'$, $y = y'$. Hence $[\,,]$ maps $P_a \times Q_a$ homeomorphically onto its image $[P_a, Q_a]$. Moreover, if $z \in A$ is close enough to a then $[z, a] \in P_a$, $[a, z] \in Q_a$, and $z = [[z, a], [a, z]]$ by (1.8). Therefore $[P_a, Q_a]$ is a neighborhood of a. $\square$

It is useful to redefine laminations in terms of local coordinates. If $\mathcal{L} \subset M$ is an immersed k-dimensional submanifold then a *lamination chart* at $x \in \mathcal{L}$ is an embedding $h: A \times D^k \to M$ such that:

(a) $A \subset D^{n-k}$ is compact;

(b) $h(A \times D^k)$ is a neighborhood of x in $\mathcal{L}$;

(c) $h(a \times D^k) \subset \mathcal{L}(h(a, 0))$ for all $a \in A$;

(d) $h_a = h \mid a \times D^k$ is a smooth embedding, and $(a, \xi) \to Th_a(\xi)$ is continuous on $A \times TD^k$.

(1.9) THEOREM. *An immersed submanifold $\mathcal{L}$ is a lamination if and only if there is a lamination chart at each $x \in \mathcal{L}$.*

PROOF. For necessity take a smooth embedding ϕ: $D^{n-k} \times D^k \to M$ with $\phi(0, 0) = x$ so that the lamination $\mathfrak{N}$ with leaves $\phi(\operatorname{int} D^{n-k} \times t)$ is transverse to $\mathcal{L}$. Then apply (1.7). Part (d) of the definition is clear from the proof of (1.6).

Conversely, suppose $\mathcal{L}$ is covered by lamination charts. Then $\mathcal{L}$ is locally compact, hence locally closed. At each $x \in \mathcal{L}$ we can find a lamination chart h and compact $A^1 \subset A^2 \subset A^3 = A$ so that, with $B^i = h(A^i \times \frac{i}{3}D^k)$, B^i is a neighborhood of B^{i-1} in $\mathcal{L}$ and B^1 is a neighborhood of x in $\mathcal{L}$. It is then routine to find a family $\{h_m$: $m \in \mathbf{Z}^+\}$ of such charts with $\{B_m^3\}$ locally finite and $\mathcal{L} = \cup B_m^1$. We construct inductively disk families X_m through neighborhoods V_m of $\cup_{j=1}^m B_j^1$ with $X_m(x) \subset \mathcal{L}(x)$ if $x \in \mathcal{L} \cap V_m$. For the inductive step let Y be a disk family through B_{m+1}^2 with $Y(h_{m+1}(a, t)) \subset h_{m+1}(a \times D^k)$. Extend this to an open neighborhood of B_{m+1}^2 by (1.2), and then restrict to an open $U \supset B_{m+1}^1$ such that $U \cap \mathcal{L} \subset B_{m+1}^2$. Hence $Y(x) \subset \mathcal{L}(x)$ for $x \in U \cap \mathcal{L}$. Shrinking U, we may assume (1.3) is applicable to X_m and Y restricted to $V_m \cap U$, with $F = T_0 X_m^\perp$. We take smaller neighborhoods U^1 of B_{m+1}^1, V_m^1 of $\cup B_j^1$, so that $V_m^1 \backslash U = V_m \backslash U$ and $K = \mathrm{Cl}(U^1 \cap V_m^1) \subset U \cap V_m$. Now shrink Y so that $Y(x) \subset X_m(x)$ on a neighborhood N of $K \cap \mathcal{L}$ in $\mathcal{L}$. Combining the isotopy of (1.3) with a bump function vanishing off N we modify Y so that $Y(x) = X_m(x)$ for $x \in U^1 \cap V_m^1$ and we still have $Y(x) \subset \mathcal{L}(x)$ for all $x \in U \cap \mathcal{L}$. Now set $V_{m+1} = V_m^1 \cup U^1$ and apply (1.4) to X_m and Y to define X_{m+1}.

By local finiteness $V = \lim V_m$ is a neighborhood of $\mathcal{L}$ and $X = \lim X_m$ is a disk family through V which restricts to a plaquation of $\mathcal{L}$. $\square$

Smoothness. We say a disk family X through an open A, parameterized by w: $E(\rho) \to F$, is C^r, $r \geq 1$, iff $E \oplus F$ is a C^r splitting of $TM \mid A$ and the maps ρ, w are C^r. A disk family through a nonopen set is C^r iff it extends to a C^r disk family on a neighborhood. A lamination $\mathcal{L}$ is C^r iff it has a C^r plaquation, and a lamination chart $h: A \times D^k \to M$ is C^r iff it and its inverse are C^r (that is, they extend to be C^r as maps on open neighborhoods). For C^r disk families X and laminations $\mathcal{L}$ the subbundles $T_0 X$ and $T\mathcal{L}$ are generally only C^{r-1}, but with this exception the obvious C^r versions of all the results in this section are true. The proofs are essentially the same, taking due care to extend disk families to open sets, and replacing $T_0 X$, $T\mathcal{L}$ by C^r approximations.

For C^r laminations we only demand that a plaquation extend to an open set. In fact, a C^r lamination extends to a C^r lamination of an open set—that is, a C^r foliation. This is clear from the proof of (1.9) as soon as we show that C^r lamination charts extend to C^r embeddings of open sets. But this follows from the following simple consequence of the inverse function theorem.

(1.10) LEMMA. *If $K \subset M$ is compact, f: $K \to N$ is C^r, and Tf: $TM|K \to TN$ is injective then f extends to a C^r embedding of some neighborhood of K.* □

Example (8.1) shows that laminations do not generally extend to open sets, even though, by (1.2), their plaquations do.

We include here one more construction with disk families which requires some smoothness.

(1.11) LEMMA. *Suppose X, Y are disk families through A with Y smooth and $T_0X \cap T_0Y = 0$. Then there is a disk family Z through A such that each $Z(a)$ is open in $\bigcup\{Y(b)$: $b \in X(a)\}$, and Z is C^r if X and Y are C^r.*

PROOF. Parameterize X, Y by ϕ_X: $E_X(\rho_X) \to TM|A$, ϕ_Y: $E_Y(\rho_Y) \to TM|A$, and let P_Y: $TM|A \to E_Y$ be the corresponding projection. By (1.1) we may assume $E_X \cap E_Y = 0$. For $\xi_X \in E_{Xa}$, $\xi_Y \in E_{Ya}$ let $b = \exp \phi_X \xi_X$, let H: $TM(\delta)_a \to TM_b$ be the map described in the proof of (1.6), and define $\psi(\xi_X + \xi_Y) = H^{-1}\phi_Y P_Y H \xi_Y$. It is straightforward to check, using pullbacks as in (1.6), that ψ is an FC^1 embedding of some neighborhood of the zero section in $E_X \oplus E_Y$. Then Z is defined by applying (1.1) to ψ. □

Topologies. There are two reasonable topologies to put on FC^1 (respectively C^r) bundle maps $E \supset U \xrightarrow{\phi} F$: the weak topology, defined by uniform convergence of ϕ and $FD\phi$ (respectively, the r-jet of ϕ) on compacta, and the corresponding strong or Whitney topology. For details see [**H**, Chapter 2]. We topologize those disk families which can be parameterized by fixed $E(\rho)$ and F accordingly.

2. Tubular families. A lamination W is *transversely zero dimensional* iff for all submanifolds T transverse to W and of complementary dimension, $T \cap W$ is zero dimensional as a subset of T. From the proof of (1.9) this is equivalent to requiring a cover of W by lamination charts h: $A \times D^k \to M$ with $\dim A = 0$. We would like to regard such laminations as generalized submanifolds, and the object of this section is to establish some submanifold-like properties. The first is simple:

(2.1) LEMMA. *If W_1, W_2 are transversely zero-dimensional laminations and $W_1 \pitchfork W_2$, then $W_1 \cap W_2$ is transversely zero dimensional.*

PROOF. Take a submanifold T meeting $W_1 \cap W_2$ transversely with complementary dimension, and let $\mathfrak{N}_i = T \cap W_i$. Then $\mathfrak{N}_1$, $\mathfrak{N}_2$ are transverse and have complementary dimensions in T so we can use canonical coordinates. Thus each $x \in \mathfrak{N}_1 \cap \mathfrak{N}_2$ has a neighborhood homeomorphic to $A_1 \times A_2$ where A_1 is a neighborhood of x in

$\mathfrak{N}_1(x) \cap \mathfrak{N}_2$ and A_2 is a neighborhood of x in $\mathfrak{N}_1 \cap \mathfrak{N}_2(x)$. But $\mathfrak{N}_1(x) \cap \mathfrak{N}_2 = (W_1(x) \cap T) \cap W_2$ which, since W_2 is transversely zero dimensional, is zero dimensional. Hence A_1 and, similarly, A_2 have dimension zero, so $A_1 \times A_2$ has dimension zero. Therefore $\mathfrak{N}_1 \cap \mathfrak{N}_2 = T \cap (W_1 \cap W_2)$ is zero dimensional. $\square$

We need a generalization of the notion of tubular neighborhood. We say $\mathfrak{L}$ is a *tubular family* for the lamination W iff $\mathfrak{L}$ is a lamination of a neighborhood of W in M which is transverse to W and has complementary dimension. The following construction of tubular families, with control, will be used repeatedly. We introduce the notation $\mathfrak{L} \leqslant \mathfrak{N}$ to mean $\mathfrak{L}(x) \subset \mathfrak{N}(x)$ for all $x \in \mathfrak{L} \cap \mathfrak{N}$.

(2.2) THEOREM. *Suppose $1 \leqslant r \leqslant \infty$. Suppose W is a transversely zero-dimensional lamination and $\mathfrak{T}$, $\mathfrak{L}_0$ are C^r laminations transverse to W with $\mathfrak{L}_0$ open in M,* $\dim \mathfrak{L}_0 = \operatorname{codim} W$, *and $\mathfrak{L}_0 \leqslant \mathfrak{T}$. If $C_0 \subset \mathfrak{L}_0 \cap W$ and $C \subset \mathfrak{T} \cap W$ are closed in W then there are open sets $V \supset C$, $V_0 \supset C_0$ and a C^r tubular family $\mathfrak{L}$ for W with $\mathfrak{L} \leqslant \mathfrak{L}_0 \cap V_0$ and $\mathfrak{L} \leqslant \mathfrak{T} \cap V$.*

PROOF. Replacing M by a suitable neighborhood of W we may assume W is closed. We first find open $V \supset C$, $V_0 \supset C_0$, $U \supset W \cup V \cup V_0$ and a C^r disk family X through U with

(a) $X(x) \pitchfork_x W(x)$ for $x \in W$,

(b) $X(x) \subset \mathfrak{L}_0(x)$ for $x \in V_0 \cap \mathfrak{L}_0$,

(c) $X(x) \subset \mathfrak{T}(x)$ for $x \in V \cap \mathfrak{T}$.

We start with $V_0 = \mathfrak{L}_0$ and X_0 a C^r plaquation of $\mathfrak{L}_0$. By definition a plaquation of $\mathfrak{T}$ extends to a C^r disk family X_T through some V, and by (1.2) a plaquation of W extends to a disk family X_W through some U. We shall shrink V, V_0, U repeatedly during the construction of X. Note that X_0 satisfies (a)–(c) for $x \in V_0$; we need to extend a restriction of X_0 to W. For $\alpha = 0, T, W$ let ϕ_α: $E_\alpha(\rho_\alpha) \to E_\alpha \oplus F_\alpha$ parameterize X_α. Shrinking V, we may assume $T_0X_T + T_0X_W | V = TV$. Since $T_0X_T = \operatorname{Im} FD_0\phi_T$ we may find a C^r $E_1 \subset E_T$ so that $FD_0\phi_T(E_1) \oplus T_0X_W | V = TV$, and let X_1 be the disk family parameterized by $\phi_T | E_1(\rho_T)$. Then (a),(c) hold for X_1. Using (1.3) we can isotope $X_1 | V \cap V_0$ to $X_0 | V \cap V_0$, maintaining (a),(c) at each stage. Using a bump function and then shrinking V and V_0 we may assume $X_1 = X_0$ on $V \cap V_0$ and (a),(c) still hold for X_1. By (1.4) we can glue X_0, X_1 together to obtain X_2 through $V \cup V_0$, satisfying (a)–(c). Now take a C^r complement E_3 to T_0X_W and define X_3 through U using the inclusion $E_3(\rho_3) \to TU$ as parameterization. Then (a) is satisfied and, as above, we can adjust X_3 to agree with X_2 on a shrunken $V \cup V_0$, defining X.

We now concentrate on the image of a lamination chart h: $A \times D^k \to M$ for W, so $\dim A = 0$. Select compact $A^i \subset A$, $1 \leqslant i \leqslant 4$, so that, defining $B^i = h(A^i \times \frac{i}{4}D^k)$, each B^{i+1} is a neighborhood of B^i in W. For $a \in A$ we write $B^i_a = h(a \times \frac{i}{4}D^k)$. Select open $V^i \supset C$ for $1 \leqslant i \leqslant 4$ so that $V^4 = V$ and $V^{i+1} \supset \operatorname{Cl} V^i$ and, similarly, $V_0^i \supset C_0$. Since $B^4 \cap C \subset \mathfrak{T}$ is compact we may shrink V^2 and find a neighborhood K of B^4 so that $K \cap \operatorname{Cl} V^2 \cap \mathfrak{T}$ is compact. Take $\delta > 0$ with $\delta < d(V^i \cap B^4, \partial V^{i+1})$, $\delta < d(V_0^i \cap B^4, \partial V_0^{i+1})$ for $i = 1, 2, 3$.

Fix $a \in A$. We next shrink X to X^1, parameterized by $\phi^1: E^1(\rho^1) \to TU$, so that:

(d) The restriction of $\exp \circ \phi^1$ to $E^1(\rho^1) \mid B_a^4$ is an embedding;

(e) $X^1(x) \subset \mathfrak{T}(x) \cap V^3$ if $x \in B_a^2$ and $X^1(x)$ meets $\mathfrak{T} \cap V^2$;

(f) $X^1(x) \subset \mathfrak{L}_0(x) \cap V^3$ if $x \in B_a^2$ and $X^1(x)$ meets $\mathfrak{L}_0 \cap V^2$.

We start by using (1.10) to shrink X to X^4 satisfying (d). Applying (1.11) to X^4 and a plaquation for $\mathfrak{T} \cap W \cap V^4$ produces a disk family Z through $\mathfrak{T} \cap W \cap V^4$ with $Z(x) \subset \bigcup\{X^4(t): t \in \mathfrak{T}(x) \cap W(x) \cap V^4\}$, so $Z(x) \subset \mathfrak{T}(x)$ by (c). We may assume $Z(x) \cap W \subset B_a^4$ if $x \in B_a^3 \cap \mathfrak{T} \cap V^4$, and then find a plaquation X_T for $\mathfrak{T}$ with $X_T(x) \subset Z(x)$ for $x \in \mathfrak{T} \cap W \cap V^3$. By (1.8) and compactness of $K \cap \mathrm{Cl}\, V^2 \cap \mathfrak{T}$ we can reduce δ so that, for $z \in K \cap V^2$ and $d(y, z) < \delta$, we have $y \in X_T(z) \Leftrightarrow z \in X_T(y)$. Shrink X^4 to X^3 so that $X^3(z) \subset X_T(z)$ for $z \in V^2 \cap \mathfrak{T}$. Using (1.6) we shrink X^3 to X^2 so that, for some neighborhood K^2 of B_a^2 in K, if $z \in K^2$ then $X^2(z)$ meets B_a^3 in some y with $d(z, y) < \delta$. Finally shrink X^2 to X^1 so that each $X^1(x)$ has diameter less than $d(B_a^2, \partial K^2) < \delta$.

To verify (e) take $x \in B_a^2$, $z \in X^1(x) \cap \mathfrak{T} \cap V^2$. Then $z \in K^2$ so $X^2(z)$ meets B_a^3 at y with $d(z, y) < \delta$. Since $y \in X^2(z) \subset X_T(z)$ we have

$$z \in X_T(y) \subset \bigcup \{X^4(t): t \in \mathfrak{T}(y) \cap B_a^4\};$$

pick such t. By (d) for X^4 we have $t = x$, so $X^1(x) \subset \mathfrak{T}(x)$. Since $X^1(x)$ meets V^2 we also have $X^1(x) \subset V^3$. The argument for (f) is similar.

Now let $X_a = X^1 \mid B_a^4$. By (d) there is a lamination $\mathfrak{P}_a$ of a neighborhood of B_a^4 whose leaves are int $X_a(t)$, so there is an open neighborhood N_a of a in A such that $B_b^3 \subset \mathfrak{P}_a$ if $b \in N_a$. Since A is compact and zero dimensional and $\{N_a: a \in A\}$ is an open cover, we can write A as a finite disjoint union of compact sets A_j with each A_j in some N_{a_j}. Then the disjoint compacta $h(A_j \times \frac{3}{4}D^k)$ have open neighborhoods $U_j \subset \mathfrak{P}_{a_j}$ with disjoint closures, so $\tilde{\mathfrak{L}} = \bigcup(U_j \cap \mathfrak{P}_{a_j})$ is a smooth lamination of a neighborhood of B^3. By (e), (f) we have $\tilde{\mathfrak{L}}(x) \subset \mathfrak{T}(x) \cap V^3$ if $x \in B^2 \cap V^2$ and $\tilde{\mathfrak{L}}(x) \subset \mathfrak{L}_0(x) \cap V_0^3$ if $x \in B^2 \cap V_0^2$. Now $\tilde{V} = \bigcup\{\tilde{\mathfrak{L}}(x): x \in V^1\}$ is open and $\tilde{V} \subset V^2$ since each $\tilde{\mathfrak{L}}(x)$ has diameter less than $d(V^1, \partial V^2)$. Hence, if $\mathfrak{L}$ is obtained from $\tilde{\mathfrak{L}}$ by restricting to a neighborhood of B^1, we have $\mathfrak{L} \leqslant \mathfrak{T} \cap \tilde{V}$, and similarly $\mathfrak{L} \leqslant \mathfrak{L}_0 \cap \tilde{V}$.

The theorem is proved by finding a countable collection h_m of lamination charts so that the corresponding B_m^1 cover W and the B_m^4 are locally finite. We extend the lamination $\mathfrak{L}$ chart by chart, using (1.4) to patch the pieces together. The constructions of X and $\mathfrak{L}$ at each stage can be localized in a small neighborhood of B_m^4 so that, by local finiteness, we still have a lamination and the desired neighborhoods V, V_0 in the limit. □

We make two observations which will be useful later. First we note that, although many choices are made, only finitely many are needed to construct a compact piece of $\mathfrak{L}$, and the steps in the construction either work uniformly for small perturbations of the initial data or are given by simple formulas (for example, the isotopies given by (1.3) are in this class). A formal statement is:

(2.3) Suppose, in (2.2), that $\mathfrak{T}$ is also open, that C_0 and C are compact with compact neighborhoods $K_0 \subset \mathfrak{L}_0$, $K \subset T$. Suppose plaquations X_0 for $\mathfrak{L}_0$, X_T for $\mathfrak{T}$

are given, and $A \subset W$ is compact. Then a plaquation X for $\mathfrak{L}$ can be constructed so that $X|A$ depends continuously, in the uniform C^r topologies, on $X_0|K_0$ and $X_T|K$. $\square$

The other observation is that in the construction of X we only required smoothness in order to apply (1.10) and (1.11) to establish (d)–(f). If we could establish them without smoothness we could still construct $\mathfrak{L}$, with each $\mathfrak{L}(x)$ contained in some $X(y)$. This is not always possible: see Example (8.1). However, if $\mathfrak{L}_0$ is a tubular family for W we can find such X, and hence we can shrink it:

(2.4) Suppose $\mathfrak{L}_0$ is a tubular family for the transversely zero-dimensional lamination W and ε: $W \to R^+$ is continuous. Then there is a tubular family $\mathfrak{L} \leqslant \mathfrak{L}_0$ for W such that each leaf meets W and diam $\mathfrak{L}(x) < \varepsilon(x)$ for $x \in W$. $\square$

3. Semi-invariant tubular families. We shall reinterpret Robinson's analysis of neighborhoods of basic sets [**R**] in light of the last section. First we review the basic facts about basic sets. An excellent summary is [**B**].

Henceforth f shall denote a diffeomorphism of M. A set $\Lambda \subset M$ is *invariant* iff $f\Lambda = \Lambda$, and a compact invariant Λ is *hyperbolic* iff $TM|\Lambda$ has a Tf invariant splitting as $E^- \oplus E^+$ such that, for some $\lambda < 1$, $c > 0$, and all $j \geqslant 0$,

$$\|Tf^{\pm j}|E^{\pm}\| \leqslant C\lambda^j.$$

We shall always assume the metric has been adjusted by Mather's theorem [**HP**, 3.1] so that $C = 1$. The stable manifold theorem [**HP**] yields unique disk families $X_\alpha^{\pm}$ through Λ for all small α, parameterized as $E^{\pm}(\alpha) \to E^{\mp}$, which are semi-invariant: $f^{\pm 1}X_\alpha^{\pm}(x) \subset X_\alpha^{\pm}(f^{\pm 1}x)$. These disk families are in fact coherent, so the interiors of the disks patch together to define immersed submanifolds $W_\alpha^{\pm}$. A hyperbolic set Λ is a *basic set* for f iff it satisfies

(a) local product structure: $W_\alpha^- \cap W_\alpha^+ = \Lambda$ for some $\alpha > 0$; and

(b) transitivity: $f|\Lambda$ has a dense orbit.

Suppose Λ is a basic set, $\gamma < \beta < \alpha$, $x \in \Lambda$, and $z \in X_\gamma^+(x)$. For α small enough an adaptation of [**R**, 4.2] shows that $\cup\{X_\beta^+(t): t \in X_\alpha^-(x) \cap \Lambda\}$ is a neighborhood of z in W_α^+. So we can use the X_β^+ to construct lamination charts, showing that W_α^+ is a lamination. Also, W_α^+ is transversely zero dimensional iff, for each $x \in \Lambda$, $W_\alpha^-(x) \cap \Lambda$ is zero dimensional. By transitivity it is sufficient to check this for one $x \in \Lambda$.

For $V \subset M$ we define

$$I^{\pm}(V) = \cap\left\{f^{\mp k}V: k \geqslant 0\right\}, \qquad I(V) = I^-(V) \cap I^+(V).$$

These are the maximal semi-invariant, respectively invariant, subsets of V. If Λ is basic and $\alpha > 0$ then from [**B**] we have $I^{\pm}(V) \subset W_\alpha^{\pm}$ for any sufficiently small neighborhood of Λ. Thus such V is an *isolating neighborhood* of Λ; that is, $I(V) = \Lambda$.

We need more structure for isolating neighborhoods. We let F_β^+ be the fundamental domain $\mathrm{Cl}(W_\beta^+ \setminus fW_\beta^+)$. From [**HPPS**] or [**R**] we have, for all sufficiently small β,

$$(3.1) \qquad F_\beta^+ \cap f^2\mathrm{Cl}(W_\beta^+) = \varnothing, \qquad \mathrm{Cl}(W_\beta^+)\setminus\Lambda = \cup\left\{f^kF_\beta^+ : k \geqslant 0\right\}.$$

If U is a neighborhood of $\mathrm{Cl}(W_\beta^+)$ and N is a neighborhood of F_β^+ we set

$$V_0 = \{f^k x \colon k \geq 0, x \in N, \text{ and } f^j x \in U \text{ if } 0 \leq j \leq k\}, \tag{3.2}$$
$$V = \operatorname{int}(V_0 \cup I^-(U)).$$

For given $\alpha > 0$ and all sufficiently small β, U, N we have

(3.3) (a) $\mathrm{Cl}\, W_\beta^+ \subset V$,

(b) $V \setminus V_0 \subset W_\alpha^-$, $V_0 \cap fW_\alpha^- = \varnothing$,

(c) $I^\pm(V) \subset W_\alpha^\pm$,

(d) $V \setminus fV \subset N$, $fV \cap N \subset fN$.

Here (a) is in [**R**, §4]. For U and β small enough $I^\pm(U) \subset W_\alpha^\pm$, which implies the first half of (b) and (c). The second half of (b) follows if $N \cap fW_\alpha^- = \varnothing$. The first half of (d) is clear, and the second half follows because $fV \setminus fN \subset fV \setminus f(V \setminus fV) \subset f^2V \subset f^2U$, which misses N for U, N small.

We now have the main result of this section.

(3.4) **Theorem.** *Suppose Λ is a basic set, $\alpha > 0$, and W_α^+ is transversely zero dimensional. Then there is a tubular family $\mathfrak{L}^-$ for W_α^+ with $\mathfrak{L}^- \leq W_\alpha^-$ which is semi-invariant: $f\mathfrak{L}^- \geq \mathfrak{L}^-$. Moreover, $\mathfrak{L}^- \setminus W_\alpha^-$ is C^r if f is C^r.*

Proof. We shall construct $\mathfrak{L}^-$ for some W_β^+ where $\beta < \alpha$ satisfies (3.1), (3.3), and then some $f^{-p}\mathfrak{L}^-$ will work for W_α^+.

Let $\mathfrak{T}_0$ be a C^r tubular family for W_α^+ given by (2.2) and take open $N_0 \subset U \subset \mathfrak{T}_0$ as in (3.3). By (3.1) we may shrink N_0 so that $P_0 = N_0 \cap fN_0$ is disjoint from $f^{-1}P_0$, so $\mathfrak{T}_1 = (\mathfrak{T}_0 \cap P_0) \cup f^{-1}(\mathfrak{T}_0 \cap P_0)$ is a lamination. Take an open neighborhood P of $F_\beta^+ \cap fF_\beta^+$ with $\overline{P} \subset P_0$, and a neighborhood Y of $F_\beta^+ \setminus (P \cup f^{-1}P)$ so small that $Y \subset N_0$ and $F_\beta^+ \cap (fY \cup f^{-1}Y) = \varnothing$. From (2.2) we have a C^r tubular family $\mathfrak{T}_2$ for $Y \cap W_\beta^+$ with $\mathfrak{T}_2 \subset Y$ and $\mathfrak{T}_2 \leq \mathfrak{T}_1 | Q$ where Q is a neighborhood in Y of $Y \cap (\overline{P} \cup f^{-1}\overline{P})$. Hence $\mathfrak{T}_3 = \mathfrak{T}_2 \cup \mathfrak{T}_1 | (P \cup f^{-1}P)$ is a C^r lamination of an open $N \supset F_\beta^+$, and clearly $N \cap fN = P$ and $\mathfrak{T}_3 \cap P = f\mathfrak{T}_3 \cap P = \mathfrak{T}_0 \cap P$. If V is the open set defined by (3.2) for N we set $\mathfrak{L}_0^- = \mathfrak{T}_3 \cup (\mathfrak{T}_0 \cap V \cap fV)$. Remembering (3.3)(d) we see that $\mathfrak{L}_0^-$ is a C^r lamination of V and $f\mathfrak{L}_0^- \geq \mathfrak{L}_0^- \cap N$.

We now define $\mathfrak{L}_k^-$ recursively by $\mathfrak{L}_{k+1}^- = (f\mathfrak{L}_k \cap V) \cup (\mathfrak{L}_k \cap N)$. Again using (3.3)(d) we see that this is well defined if $f\mathfrak{L}_k^- \geq \mathfrak{L}_k^- \cap N$ and that we still have $f\mathfrak{L}_{k+1}^- \geq \mathfrak{L}_{k+1}^- \cap N$. Each $\mathfrak{L}_k^-$ is a C^r tubular family for W_β^+. By induction one checks $\mathfrak{L}_k^-(x) = \mathfrak{L}_m^-(x)$ and $\mathfrak{L}_k^-(x) \subset V_0$ for $k, m > K$ and $x \in V \setminus \bigcap \{f^j V \colon j \leq K\}$, where V_0 is given by (3.2). Hence $\mathfrak{L}_*^- = \lim(\mathfrak{L}_k^- \cap V_0)$ exists as a C^r tubular family for $W_\beta^+ \setminus \Lambda$, and $\mathfrak{L}^- = \mathfrak{L}_*^- \cup (W_\alpha^- \cap V)$ is a well-defined immersed submanifold with $f\mathfrak{L}^- \geq \mathfrak{L}^-$. We need lamination charts for $\mathfrak{L}^-$.

Restrict a plaquation for $\mathfrak{L}_0^-$ to $\mathrm{Cl}\, W_\beta^+$ giving a disk family X_0^-. After shrinking X_0^-, Robinson's abstract stable manifold theorem [**R**, 3.1] applies to give a sequence X_k^- of disk families converging FC^1 to a disk family X^- with $X^-(x) \subset W_\alpha^-(x)$ if $x \in \Lambda$. Comparing the definitions of X_k^- and $\mathfrak{L}_k^-$ it is clear that $X_k^-(x) \subset \mathfrak{L}_k^-(x)$ for all x, so X^- defines lamination charts for $\mathfrak{L}^-$ covering Λ. Using invariance we then find lamination charts at points of $W_\alpha^- \cap V$. Of course there is no problem with lamination charts in V_0. $\square$

We now list several refinements of (3.4) for later use. The first is clear from (2.2).

(3.5) If a C^r lamination $\mathfrak{T}$ is given transverse to W_α^+ with $f\mathfrak{T} \geq \mathfrak{T}$ and $K \subset \mathfrak{T} \cap F_\beta^+$ is compact, for some $\beta \leq \alpha$, then we may assume $\mathfrak{L}^- \leq \mathfrak{T} \cap V$ where V is some neighborhood of K. □

Next we just state explicitly the convergence properties of the $\mathfrak{L}_k^-$.

(3.6) There are C^r tubular families $\mathfrak{L}_k^-$ for W_α^- with $\mathfrak{L}_k^- = \mathfrak{L}^-$ as sets so that

(a) if K is a compact neighborhood of Λ then, for all sufficiently large k, if $x \in \mathfrak{L}^- \setminus K$ then $\mathfrak{L}_k^-(x) = \mathfrak{L}^-(x)$;

(b) there are plaquations X_k^- for $\mathfrak{L}_k^-$, X^- for $\mathfrak{L}^-$ with $X_k^- \to X^-$ in the weak FC^1 topology. □

We next consider perturbations of f. For g in some neighborhood of f there is a neighborhood V of Λ_f so that $\Lambda_g = I(V, g)$ is hyperbolic (see **[B]**). From **[R]** the unstable disks $X_g^{\pm}(x)$ for $x \in \Lambda_g$ can be chosen FC^1 continuously in g. Finally consider V_g defined by (3.2) for g, where N and U are independent of g. The only part of (3.3) not satisfied for V_g is (a). This was proved in **[R]** using a disk family through $W_{\alpha f}^+$ depending FC^1 continuously on g, so that in fact (a) can be replaced by

(a′) There is a compact set K with $\operatorname{Cl} W_{\beta g}^+ \subset \operatorname{int} K \subset K \subset V_g$ for g near f.

Putting all this together with (2.3) and the permanence part of **[R**, 3.1**]** we have

(3.7) There is a compact K and a neighborhood $\mathfrak{N}$ of f in $\operatorname{Diff}^r(M)$ such that for each $g \in \mathfrak{N}$ there is a lamination $\mathfrak{L}_g^-$ as in (3.4) with a plaquation X_g^- such that $\Lambda_g \subset \operatorname{int} K \subset K \subset \mathfrak{L}_g$ for $g \in \mathfrak{N}$ and $X_g^- | K$ depends continuously on g. This is true in the FC^1 topology on X_g^-, and in the C^r topology on X_g^- away from $W_{\alpha g}^-$. □

Finally we consider smoothness. We say $f^{\pm 1}$ *strongly contracts* $E^{\pm}$ iff

$$\|Tf | E^+\| \cdot \|Tf^{-1} | E^-\| \cdot \|Tf^{\mp 1} | E^{\pm}\| < 1. \tag{3.8}$$

(3.9) If f is C^2 and f^{-1} strongly contracts E^- then $\mathfrak{L}^-$ is smooth.

PROOF. We need a version of the C^r section theorem **[HPS**, 3.5**]** when the base is not overflowing, but the techniques for this are in **[R**, 3.2**]**. We then proceed as in **[HP**, 6.3**]** to deduce smoothness of $T\mathfrak{L}^-$. From the Frobenius theorem we deduce smoothness of $\mathfrak{L}^-$. □

4. Markov neighborhoods. Bowen refined the canonical coordinates for a basic set to obtain Markov partitions, which have proved to be extremely useful (again see **[B]**). In this section we use the tubular families constructed in §3 to extend this structure to a neighborhood of a zero-dimensional basic set.

First some definitions. If $\mathfrak{L}^{\pm}$ are transverse laminations of open sets of complementary dimensions then the (X^-, X^+) canonical coordinates associated to plaquations $X^{\pm}$ of $\mathfrak{L}^{\pm}$ by (1.7) will be called the $\mathfrak{L}^{\pm}$ canonical coordinates. A closed connected set R is called a *rectangle* iff $[x, y]$ is defined and lies in R whenever $x, y \in R$. A closed set is called *rectangular* iff its components are rectangles. For a rectangular set Q and $x \in Q$ we let $Q^{\pm}(x)$ be the component of x in $\mathfrak{L}^{\pm}(x) \cap Q$ and set $\partial Q^{\pm}(x) =$ boundary of $Q^{\pm}(x)$ in $\mathfrak{L}^{\pm}(x)$ and $\partial^{\pm} Q = \{x \colon Q^{\pm}(x) \subset \partial Q\}$. The following is left to the reader.

(4.1) (a) The intersection of two rectangular sets is rectangular.

(b) If R is a rectangle and $x \in R$ then $R = [R^-(x), R^+(x)]$.

(c) If R is a rectangle then $\partial R = \partial^- R \cup \partial^+ R$ and $\partial^\pm R = \bigcup\{\partial R^\mp(x)\colon x \in R\}$. $\square$

We generally apply lamination terminology to $Q^\pm$ and $\partial^\pm Q$. Note that $\operatorname{int} Q^\pm$, defined by $\operatorname{int} Q^\pm(x) = Q^\pm(x)\backslash \partial Q^\pm(x)$, is a lamination.

Now suppose Λ is a basic set for f. We say a compact neighborhood Q of Λ with $I^\pm(Q) \subset W_\alpha^\pm$ is a *Markov neighborhood* iff it has finitely many components, is rectangular with respect to some $\mathfrak{L}^\pm$, and satisfies $f^{\pm 1}Q^\pm \leqslant \operatorname{int} Q^\pm$.

(4.2) THEOREM. *If Λ is a zero-dimensional basic set and $\mathfrak{L}^\pm$ are semi-invariant tubular families for $W_\alpha^\mp$ then Λ has a Markov neighborhood Q with $Q^\pm \leqslant \mathfrak{L}^\pm$.*

PROOF. The construction is in three stages. First, find a rectangular neighborhood; second, improve this to a Markov neighborhood for some iterate f^m, with some extra properties; third, proceed by downward induction on m to produce a Markov neighborhood for f.

Step 1. Restricted to a neighborhood of Λ we have $\mathfrak{L}^- \pitchfork \mathfrak{L}^+$ so canonical coordinates are defined. By (1.7) we can find a closed neighborhood of any $x \in \Lambda$ of the form $D_x = [D_x^-, D_x^+]$ where $D_x^\pm \subset \mathfrak{L}^\pm(x)$ are disks. We assume $D_x \subset V$ given by (3.2, 3). Since Λ is zero dimensional we can find a compact neighborhood K of Λ with finitely many components, each contained in some D_x. If $C \subset D_x$ is such a component then $\partial C \cap \Lambda = \varnothing$, so there is $\eta > 0$ such that if $x^\pm \in D_x^\pm \cap \Lambda$ and $[x^-, x^+] \in C$ then, if $d(x^\pm, y^\pm) < \eta$, we have $[y^-, y^+] \in C$. Take closed neighborhoods $A^\pm$ of $D_x^\pm \cap \Lambda$ in $\mathfrak{L}^\pm(x)$ with finitely many components $A_j^\pm$ of diameter less than η. Then the union S_C of the rectangles $[A_j^-, A_k^+]$ which meet $C \cap \Lambda$ satisfies $C \cap \Lambda \subset \operatorname{int} S_C$, $S_C \subset C$. Thus $S = \bigcup_C S_C$ is a rectangular neighborhood of Λ. For η small enough we have $f^{\pm 1}S \subset \mathfrak{L}^- \cap \mathfrak{L}^+$.

Step 2. For $k \geqslant 0$ we set $S_k = \bigcap\{f^{-j}S\colon 0 \leqslant j \leqslant k\}$. We want to find m^+ so that, for $m \geqslant m^+$, S_m is a rectangular neighborhood and

(a) $f^m S_m^+ \leqslant \operatorname{int} S_k^+$ for all $k \geqslant 0$,

(b) $f^j S_m^+ \leqslant S_m^+$ for $0 \leqslant j \leqslant m$.

As a start we note that, since $f^{-1}S \subset \mathfrak{L}^- \cap \mathfrak{L}^+$ and the $\mathfrak{L}^\pm$ are semi-invariant, $f^{-1}S$ is rectangular. Using this, (4.2)(a), and induction we see that each S_k is a rectangular neighborhood of Λ. Also, if $S^+(x)$ and $fS^+(y)$ meet they lie in the same leaf of $\mathfrak{L}^+$. From this, induction, and (4.2)(c) we derive

(c) $\partial^- S_k \subset \{x \in S_k\colon f^j x \in \partial^- S$ for some j, $0 \leqslant j \leqslant k\}$.

By (3.3), $I^-(S) \subset W_\alpha^-$. Since $\mathfrak{L}^- \leqslant W_\alpha^-$, if $x \in \partial^- S \cap I^-(S)$ then $S^-(x) \subset W_\alpha^-$. But each component of S meets Λ, so for some $y \in \Lambda$, $S^+(x) \cap S^-(y) \neq \varnothing$. By local product structure, $\Lambda \cap \partial^- S \neq \varnothing$, contradicting $\Lambda \subset \operatorname{int} S$. Hence $\partial^- S \cap I^-(S) = \varnothing$, so there is $m^+ > 0$ so that

(d) $S_{-m} \cap \partial^- S = \varnothing$ for $m \geqslant m^+$, where $S_{-m} = \bigcap\{f^k S\colon 0 \leqslant k \leqslant m\} = f^m S_m$.

Now we can derive (a),(b). First we check (a), which is clearly implied by $f^m S_m \cap \partial^- S_k = \varnothing$ for all $k \geqslant 0$. So suppose $x \in f^m S_m \cap \partial^- S_k$. By (c) there is $j \geqslant 0$ such that $x \in S_j$ and $f^j x \in \partial^- S$. But $S_j \cap S_{-m} = f^{-j} S_{-m-j}$, so $f^j x \in \partial^- S \cap S_{-m-j}$, contradicting (d). For part (b), suppose x, $f^j x \in S_m$. For $0 \leqslant j \leqslant m$ this implies $x \in S_{m+j}$ so $f^m x \in S_j$. By (a), $f^m S_m^+(x) \subset S_j^+(f^m x)$. Arguing as above we conclude

$f^iS_m^+(x) \subset S$ for $0 \leqslant i \leqslant m + j$, hence $f^jS_m^+(x) \subset S_m$, so $f^jS_m^+(x) \subset S_m^+(f^j(x))$.

Repeating this argument for f^{-1} we derive the analog of (a), (b) for $m \geqslant m^-$. We take $m = \max\{m^\pm\}$, and set $T = S_{2m} = f^{-2m}S_{-2m}$. Noticing $S_{2m} \subset S_m$, (a), (b) and their analogs yield

$$(A_m^\pm) \quad f^{\pm k}T^\pm \leqslant \operatorname{int} T^\pm, \qquad m \leqslant k < 2m,$$
$$(B_m^\pm) \quad f^{\pm k}T^\pm \leqslant T^\pm, \qquad 0 < k < m.$$

These are still true after we delete the components of T which do not meet Λ. Then, by compactness, T has finitely many components.

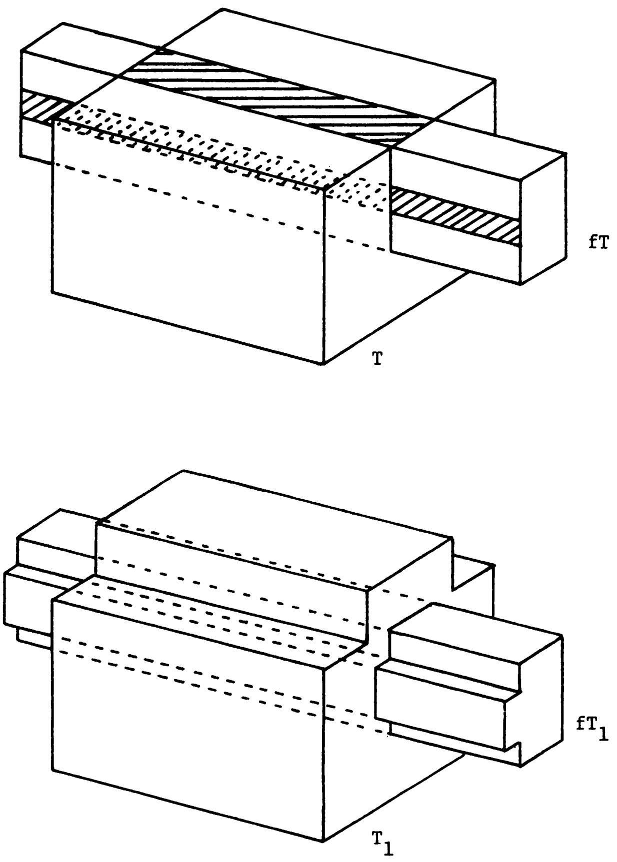


FIGURE 4.1.
Step 3 in Theorem 4.2: adding N. Here $m = 2$. P and fP are shaded in the top picture.

Step 3. We modify T, keeping finitely many components, to obtain $(A_{m-1}^\pm)$, $(B_{m-1}^\pm)$. Iterating this procedure, we finally reach $(A_1^\pm)$, and the proof is complete.

So let $P = f^{m-1}T \cap \partial^-T$. From (A_m^+) we have $f^kP \cap \partial^-T \subset f^kT \cap \partial^-T = \varnothing$ for $m \leqslant k < 2m$, and $f^kP \cap \partial^-T \subset f^{k+m-1}T \cap \partial^-T = \varnothing$ for $1 \leqslant k < m$. Also, if $y \in P$

then $f^{-m+1}y = x \in T$, so $f^{m-1}T^-(x) \supset T^-(y)$ by (B_m^-), so $T^-(y) \subset P$. Thus for each component C of T, $P_C = P \cap C$ is a product $[T^-(y), P \cap T^+(y)]$. We take a small closed neighborhood V_C of $P \cap T^+(y)$ with $T^+(y) \cup V_C$ connected and set $N_C = [T^-(y), V_C]$, $N = \bigcup N_C$, $T_1 = T \cup N$.

We claim that, for N small enough, T_1 satisfies (A_{m-1}^+), (B_{m-1}^+). Now

$$\text{(e)} \qquad f^k N \cap N = f^k N \cap \partial^- T = \varnothing, \qquad 1 \leqslant k < 2m,$$

for N small, since these are true for P. From this we check $f^k T_1 \cap \partial^- T_1 = \varnothing$ for $m \leqslant k < 2m$ and $f^k T_1^+ \leqslant T_1^+$ for $0 < k < m$. So all we need is $f^{m-1}T_1 \cap \partial^- T_1 = \varnothing$. Using (e),

$$f^{m-1}T_1 \cap \partial^- T_1 \subset \left[\partial^- T \backslash N \cap f^{m-1}T\right] \cup \left[\partial^- N \backslash T \cap f^{m-1}T\right] \cup \left[\partial^- N \cap \partial^- T \cap f^{m-1}T\right].$$

Now the last term is empty because $\partial^- T \cap f^{m-1}T = P$ misses $\partial^- N$ and the first is empty because $P \subset N$. If the middle term is nonempty for all choices of N then there is a sequence $x_n \to y \in P$ with $x_n \in f^{m-1}T \backslash T$. Because T is compact and has finitely many components we can find a component of $f^{m-1}T$ which contains both y and points arbitrarily close to y in $\mathfrak{L}^+(y) \backslash T$, contradicting (B_m^+).

Next we note that (A_m^-), (B_m^-) are still satisfied for T_1 if N is small enough. In fact, this is immediate if N is chosen so that, for $0 < k < 2m$, components of $f^{-k}T_1$ meet components of T_1 only if the corresponding components of $f^{-k}T$ and T meet. In this case the relations between $f^{-k}T_1^-$ and T_1^- readily follow from those between $f^{-k}T^-$ and T^-, using the product structure.

Similarly, (A_{m-1}^+), (B_{m-1}^+) remain satisfied if we now apply the above procedure to f^{-1} to obtain (A_{m-1}^-), (B_{m-1}^-). The procedure requires $f^{m-1}T^+ \leqslant T^+$, which follows from (A_{m-1}^+), and otherwise only uses (A_m^-), (B_m^-). So we are finished with the induction step and the proof. □

The canonical coordinates associated to a Markov neighborhood are generally not smooth, but for many purposes a smooth approximation is sufficient. We say a Markov neighborhood Q is *approximately* C^r iff f is C^r, there are C^r laminations $\tilde{\mathfrak{L}}^{\pm}$ such that Q is rectangular with respect to $\tilde{\mathfrak{L}}^{\pm}$, and, with $\tilde{Q}^{\pm} = Q \cap \tilde{\mathfrak{L}}^{\pm}$, we have

(4.3)(a) $\tilde{Q}^{\pm}(x) = Q^{\pm}(x)$ for x in a neighborhood of $\mathrm{Cl}(Q \backslash f^{\mp 1}Q)$,

(b) $\tilde{Q}^{\pm} \pitchfork Q^{\mp}$,

(c) each $\tilde{Q}^{\pm}(x)$ is a smooth manifold.

Immediate consequences of these properties are

(4.4)(a) $Q^{\pm}(x)$ is C^r diffeomorphic to $\tilde{Q}^{\pm}(y)$ for x, y in the same component of Q,

(b) int $Q^{\pm}$ is a C^r lamination off $I^{\pm}(Q)$,

(c) given $k > 0$, $\tilde{Q}^{\pm}$ can be chosen to agree with $Q^{\pm}$ off $Q \backslash f^{\mp k}Q$.

To prove (c) just apply graph transforms as in (3.4).

(4.5) Addendum to (4.2). Suppose f is C^r and the $\mathfrak{L}^{\pm}$ satisfy (3.6). Then Λ has an approximately C^r Markov neighborhood Q with $Q^{\pm} \leqslant \mathfrak{L}^{\pm}$.

Proof. The Q constructed in (4.2) satisfies (4.3)(a),(b) by virtue of (3.6). For (4.3)(c) just replace each component $[Q^-(x_k), Q^+(x_k)]$ of Q by $[P_k^-, P_k^+]$ where $P_k^{\pm} \subset \mathrm{int}\, Q^{\pm}(x_k)$ is a C^r approximation to $Q^{\pm}(x_k)$. □

If Q is a Markov neighborhood of Λ we write the components of Q as $Q_k = [Q_k^-, Q_k^+]$, where $Q_k^{\pm}$ abbreviates $Q^{\pm}(x_k)$ for some $x_k \in \Lambda \cap Q_k$, and we set $Q_*^{\pm} = \bigcup Q_k^{\pm}$. If we replace Q by $Q \cap f^{-1}Q$ we still have a Markov neighborhood, and we gain the additional property that each $Q_j \cap f^{-1}Q_k$ is connected. Thus there are well defined $f_{jk}^+ : Q_j^+ \to Q_k^+ \cap fQ_j$ and f_{jk}^-: $Q_j^- \cap f^{-1}Q_k \to Q_k^-$, whenever $Q_j \cap f^{-1}Q_k \neq \varnothing$, so that $f[x^-, x^+] = [f_{jk}^- x^-, f_{jk}^+ x^+]$ on $Q_j \cap f^{-1}Q_k$.

Now suppose g is another diffeomorphism with corresponding Λ_g, Q_{gj}, $[\,,]_g$, and $g_{jk}^{\pm}$. Then we have the following, rather standard, construction of a topological conjugacy.

(4.6) Lemma. *Suppose $Q_{gj} \cap g^{-1}Q_{gk} \neq \varnothing$ iff $Q_j \cap f^{-1}Q_k \neq \varnothing$, and*

$$h_0^{\pm} : \mathrm{Cl}\left(Q_{g*}^{\pm} \setminus g^{\pm 1}Q_g\right) \to \mathrm{Cl}\left(Q_*^{\pm} \setminus f^{\pm 1}Q\right)$$

are homeomorphisms with $h_0^{\pm} g_{jk}^{\pm} = f_{jk}^{\pm} h_0^{\pm}$ wherever defined. Then the $h_0^{\pm}$ extend to homeomorphisms $h^{\pm}$: $Q_{g*}^{\pm} \to Q_*^{\pm}$ so that, with h defined by $[x^-, x^+]_g \mapsto [h^- x^-, h^+ x^+]$, we have $hg = fh$ on $Q_g \cap g^{-1}Q_g$.

Proof. We consider only h_0^+; the argument for h_0^- follows upon interchanging f and f^{-1}, g and g^{-1}. Note that the images of the various g_{jk}^+ are disjoint and the union of their images is $Q_{g*}^+ \cap gQ_g$, and similarly for f. Hence, recalling $h_0^+ g_{jk}^+ = f_{jk}^+ h_0^+$ on ∂Q_{g*}^+, we can extend h_0^+ to

$$\bigcup \mathrm{Im}\, g_{jk}^+ \setminus g^2 Q_g \to \bigcup \mathrm{Im}\, f_{jk}^+ \setminus f^2 Q$$

by $x^+ \mapsto f_{jk}^+ h_0^+ (g_{jk}^+)^{-1} x^+$. Iterating this procedure, we eventually extend h_0^+ to a homeomorphism $Q_{g*}^+ \setminus I^-(Q_g, g) \to Q_*^+ \setminus I^-(Q, f)$ with $h_0^+ g_{jk}^+ = f_{jk}^+ h_0^+$. Then h_0^+ matches the components of the neighborhoods $g^{k+1}Q_g \cap Q_{g*}^+$ of the zero-dimensional set $I^-(Q_g, g) \cap Q_{g*}^+$ with the corresponding components for f. Such a matching converges to a homeomorphism $\tilde{h}^+ : Q_{g*}^+ \cap I^-(Q_g, g) \to Q_*^+ \cap I^-(Q, f)$, and it is easy to see that h_0^+ and $\tilde{h}^+$ fit together to define the required h^+. □

We finally have the following sharpening of the local structural stability theorem, showing that much of the structure of a Markov neighborhood is preserved under perturbation.

(4.7) Theorem. *For any sufficiently small, approximately C^r Markov neighborhood Q of Λ there is a neighborhood $\mathfrak{N}$ of f in* $\mathrm{Diff}^r(M)$ *such that, for all $g \in \mathfrak{N}$, $\Lambda_g = I(Q, g)$ is a basic set for g with an approximately C^r Markov neighborhood Q_g, satisfying*

(a) *$Q_g^{\pm}(x)$ is diffeomorphic to $Q^{\pm}(x)$ for all $x \in \Lambda_g$,*

(b) *there is a homeomorphism h: $Q_g \to Q$ with $hQ_g^{\pm}(x) = Q^{\pm}(hx)$ for all x and $hg = fh$.*

Proof. Define $Q_k^{\pm}$, etc., as before (4.6). Using (3.7) we have laminations $\mathfrak{L}_g^{\pm}$ converging to $\mathfrak{L}^{\pm}$ on Q, and the $\mathfrak{L}_g^{\pm}$ are themselves limits of smooth laminations. Pick $x_k \in Q_k \cap \Lambda$ and set $Q_{gk}^{\pm} = Q_k \cap \mathfrak{L}_g^{\pm}(x_k)$. For g close enough to f the $\mathfrak{L}_g^{\pm}$ define canonical coordinates on a neighborhood of Q and $Q_{gk} = [Q_{gk}^-, Q_{gk}^+]_g$ is a rectangle. Set $Q_g = \bigcup Q_{gk}$. For g close to f we have $\partial^{\mp} Q_g \cap g^{\pm 1}Q_g = \varnothing$ from the corresponding facts for f, so Q_g is Markov for g. For g close enough to f we have $Q_g^{\pm} \pitchfork \tilde{\mathfrak{L}}^{\mp}$, where $\tilde{\mathfrak{L}}^{\mp}$ are smooth approximations to $\mathfrak{L}^{\mp}$ and (4.3) holds. Then using $\tilde{\mathfrak{L}}^{\pm}$ canonical coordinates we see that each $Q_{gk}^{\pm}$ is diffeomorphic to $Q_k^{\pm}$, proving (a).

For (b) we shall use (4.6), so we need to assume Q small enough that each $Q_j \cap f^{-1}Q_k$ is connected. We show how to construct h_0^+; a similar argument produces h_0^-.

Projection along the fibers Q^- defines a homeomorphism ϕ of Q_{g*}^+ onto Q_*^+ which is a diffeomorphism off $I^-(Q, f)$ by (4.4). Then $f_{jk}^+ \mid \partial Q_j^+$ and $\phi \circ g_{jk}^+ \circ \phi^{-1} \mid \partial Q_j^+$ are C^1 close by (3.7) and hence are isotopic. Using the isotopy extension theorem we can modify ϕ on a small neighborhood of $\bigcup g_{jk}^+(\partial Q_{gj}^+)$ to define a homeomorphism $\tilde{\phi}$ with $\tilde{\phi} g_{jk}^+ = f_{jk}^+ \tilde{\phi}$ on ∂Q_{gj}^+. Then h_0^+ is the restriction of $\tilde{\phi}$. □

5. Tameness criteria for basic sets. A compact zero-dimensional set $\Lambda \subset M$ is *tame* (in M) iff any neighborhood of Λ contains a smaller neighborhood whose components are smoothly embedded disks. At least for dim $M \neq 4$ this is equivalent, via a theorem of Stallings **[St]**, to the standard definition, which only requires topologically embedded disks. A general discussion is in **[K]**.

Our principal interest in tameness arises from its connection with isotopies, as in (5.1)(c) below. An isotopy h_t will always be a smooth ambient isotopy, and will be called an *ε-isotopy* if $d(h_t x, h_0 x) < \varepsilon$ for all t, x, or an *ε-push* if, in addition, $h_0 = 1$. The support of h_t is $\mathrm{Cl}\{x\colon \text{for some } t,\ h_t x \neq h_0 x\}$.

(5.1) LEMMA. *Suppose $\Lambda \subset M$ is compact and zero dimensional. Then the following are equivalent.*

(a) *Λ is tame in M.*

(b) *Each $x \in \Lambda$ is contained in arbitrarily small smooth n-disks D with $\Lambda \cap \partial D = \varnothing$.*

(c) *If $\varepsilon > 0$, N is a compact submanifold of positive codimension, and V is a neighborhood of $\Lambda \cap N$, there is an ε-push h_t with support in V such that $\Lambda \cap h_1 N = \varnothing$.*

PROOF. (a) ⇒ (c): Cover Λ by disjoint n-disks D_j so that $D_j \subset V$ if $D_j \cap N \neq \varnothing$. Now construct h_t by pushing radially in each D_j, using some point in int $D_j \setminus N$ as center.

(c) ⇒ (b): Just push ∂D off Λ.

(b) ⇒ (a): Given a neighborhood V of Λ, cover Λ by finitely many n-disks $D_j \subset V$ with $\partial D_j \cap \Lambda = \varnothing$. We may assume $D_i \not\subset D_j$ for $i \neq j$. For $j > 1$ pick $x \in \operatorname{int} D_1 \setminus D_j$ and push D_j off D_1 by a radial push with x as center, producing a new disk $\tilde{D}_j$ disjoint from D_1. If this push has support in a small neighborhood of D_1 then $\partial \tilde{D}_j$ is close to $\partial D_1 \cup \partial D_j$, so $\partial \tilde{D}_j \cap \Lambda = \varnothing$. Similarly $(D_1 \cup \tilde{D}_j) \cap \Lambda = (D_1 \cup D_j) \cap \Lambda$. In this fashion push $D_2, \ldots, D_m$ off D_1, then push $\tilde{D}_3, \ldots, \tilde{D}_m$ off $\tilde{D}_2$, etc. □

The push in (c) cannot generally be made C^1 small. See Example (8.2).

(5.2) THEOREM. *Any zero-dimensional basic set Λ is tame in M.*

PROOF. Let Q be an approximately smooth Markov neighborhood of Λ. Pick a neighborhood U of some $z \in \Lambda$. We shall produce a smooth disk neighborhood of z in U whose boundary misses Λ, verifying (5.1)(b).

Start with open neighborhoods $U^{\pm}$ of z in $Q^{\pm}(z)$ with $[U^{-}, U^{+}] \subset U$ and $\partial U^{\pm} \cap \Lambda = \varnothing$. Then take compact neighborhoods $A^{\pm}$ of z in $U^{\pm}$ with $\partial A^{\pm} \cap \Lambda = \varnothing$ which lie in smooth disks in $U^{\pm}$. Let $B^{\pm}$ be compact neighborhoods of $\Lambda \cap U^{\pm} \backslash A^{\pm}$ in $U^{\pm} \backslash A^{\pm}$ and set $C^{\pm} = A^{\pm} \cup B^{\pm}$. By uniqueness of disks we can find smooth disks $D^{\pm} \subset U^{\pm} \backslash C^{\pm}$ and smooth isotopies $h_t^{\pm}$ of $U^{\pm}$ with $h_0^{\pm} = 1$ and $h_1^{\pm} A^{\pm} \subset D^{\pm}$.

Define H^{-} on $[U^{-}, U^{+}]$ by

$$H^{-}[x, y] = \left[h^{-}_{\lambda(y)}x, y\right],$$

where λ is a bump function which is 1 on A^{+} and 0 on B^{+}. Then define

$$H^{+}[x, y] = \left[x, h^{+}_{\mu(x)}y\right],$$

where μ is 1 on $h_1^{-}A^{-}$ and 0 on $C^{-} \cup h_1^{-}B^{-}$, and set $H = H^{+}H^{-}$. Notice that $H[A^{-}, A^{+}] = [h_1^{-}A^{-}, h_1^{+}A^{+}] \subset [D^{-}, D^{+}]$, while $H^{-}[C^{-}, B^{+}] = [C^{-}, B^{+}]$ and $H^{-}[B^{-}, A^{+}] = [h_1^{-}B^{-}, A^{+}]$ are not moved by H^{+}, so $H[C^{-}, C^{+}] \cap [D^{-}, D^{+}] = H[A^{-}, A^{+}]$.

If Q is smooth then H is a diffeomorphism and we obtain D as $H^{-1}\tilde{D}$ where $\tilde{D}$ is a smooth approximation to the smooth product $[D^{-}, D^{+}]$. In general we just take a smooth approximation $\tilde{Q}$ to Q which agrees with $\tilde{Q}$ on $[U^{-}, U^{+}] \backslash [C^{-}, C^{+}]$ and repeat the construction using $\tilde{Q}$ coordinates. □

This theorem is misleading since the special neighborhoods of Λ whose components are disks have no particular relation to the dynamics. Of more interest is the placement of Λ in the stable and unstable manifolds. We say Λ is *tame in* W^{+} iff for any $\alpha > 0$ there is a compact neighborhood Z^{+} of Λ in W_{α}^{+} such that, for each $x \in \Lambda$, the component $Z^{+}(x)$ of x in $Z^{+} \cap W_{\alpha}^{+}(x)$ is a smooth disk. Newhouse introduced a similar notion in **[N]** and gave an example of a zero-dimensional basic set which is wild (i.e., not tame) in W^{+}. The reader may check that tameness in W^{+} is equivalent to tameness of $\Lambda \cap \mathrm{Cl}\, W_{\beta}^{+}(x)$ in $W_{\alpha}^{+}(x)$ for all $\beta < \alpha$, $x \in \Lambda$. A stronger condition is obtained if we require Z^{+} to be strongly semi-invariant: $fZ^{+} \subset$ interior of Z^{+} in W_{α}^{+}. In this case we say Λ is *dynamically tame* in W^{+}. An easy exercise is

(5.3) Λ is tame in W^{+} if and only if it is dynamically tame in W^{+} with respect to f^{m} for some $m > 0$. □

Clearly Λ is dynamically tame in W^{+} if it has a Markov neighborhood T with each $T^{+}(x)$ a smooth disk. Conversely,

(5.4) LEMMA. *A zero-dimensional basic set Λ is dynamically tame in W^{+} if and only if for any approximately smooth Markov neighborhood Q there is a Markov neighborhood T with $T^{-}(x) = Q^{-}(x)$ and $T^{+}(x)$ a smooth disk in $Q^{+}(x)$, for all $x \in T$.*

PROOF. Suppose Λ is dynamically tame in W^{+} and Z^{+}, Q are given. Replacing Z^{+} by $f^{k}Z^{+}$ for some large k we may assume $Z^{+} \subset Q$. Using continuity and compactness we find $\delta > 0$ so that $[Q^{-}(fx), fZ^{+}(x)] \cap \partial Z^{+}(y) = \varnothing$ for $x, y \in \Lambda$ and $d(fx, y) < \delta$. Let $R = \bigcap\{f^{-j}Q\colon 0 \leqslant j \leqslant p\}$. This is a Markov neighborhood with $R^{+}(x) = Q^{+}(x)$ and $f^{p}R^{-}(x) = Q^{-}(f^{p}x)$ for all $x \in R$, and for p large enough each $fR^{-}(x)$ has diameter less than δ. For each component C of R select

$x_C \in C \cap \Lambda$ and set $V_C = \bigcup\{Z^+(y): y \in \Lambda \cap R^+(x_C)\}$. Then V_C is a neighborhood of $\Lambda \cap R^+(x_C)$ in $R^+(x_C)$ whose components are smooth disks. By choice of δ and p, $S = \bigcup_C[R^-(x_C), V_C]$ is a Markov neighborhood, and, by approximate smoothness, each $S^+(x)$ is diffeomorphic to some $Z^+(y)$. Then $T = f^pS$ is the desired neighborhood. □

Combining (5.3), (5.4) and (4.7) yields

(5.5) COROLLARY. *The properties "tame in W^+" and "dynamically tame in W^+" persist under small C^1 perturbations of f.* □

In fact, tameness greatly simplifies local conjugacy questions, allowing us to describe local conjugacy classes in terms of suitable algebraic data. Suppose Q is a Markov neighborhood of some Λ with components Q_j. Replacing Q with $fQ \cap Q$ if necessary, we may assume the $fQ_j \cap Q_k$ are connected. If $x \in \Lambda$ then $Q^{\pm}(x)$ lies in a disk of the same dimension, so is orientable. We choose such orientations consistently, using canonical coordinates, as x varies in each Q_j. Following Bowen and Franks [**BF**] we define matrices $A^{\pm}$ by $A^{\pm}_{jk} = 0$ if $fQ_j \cap Q_k = \varnothing$, and otherwise $A^{\pm}_{jk} = 1$ or -1 depending on whether f preserves or reverses the corresponding orientations on $Q^{\pm}(x)$ and $Q^{\pm}(fx)$ for $x \in Q_j \cap f^{-1}Q_k$.

This information (in fact, just the matrix of absolute values) determines $f|\Lambda$ by symbolic dynamics, but it is not sufficient to determine f on a neighborhood of Λ without tameness: consider the example of Newhouse [**N**], reproduced in Example (8.3). Unfortunately, it is not sufficient to determine f near Λ if either W^+ or W^- is one dimensional: the local diffeomorphisms in $\mathbf{R}^2$ shown in Figure 5.1 have the same $A^{\pm}$'s (using standard orientations) but cannot be locally conjugate. To see this notice that, for the fixed point p of f, one component of $Q^+(p)\backslash\{p\}$ misses Λ. However, for the fixed point q of g and either component C of $Q^+(q)\backslash\{q\}$ we have $q \in \mathrm{Cl}(C \cap \Lambda)$.

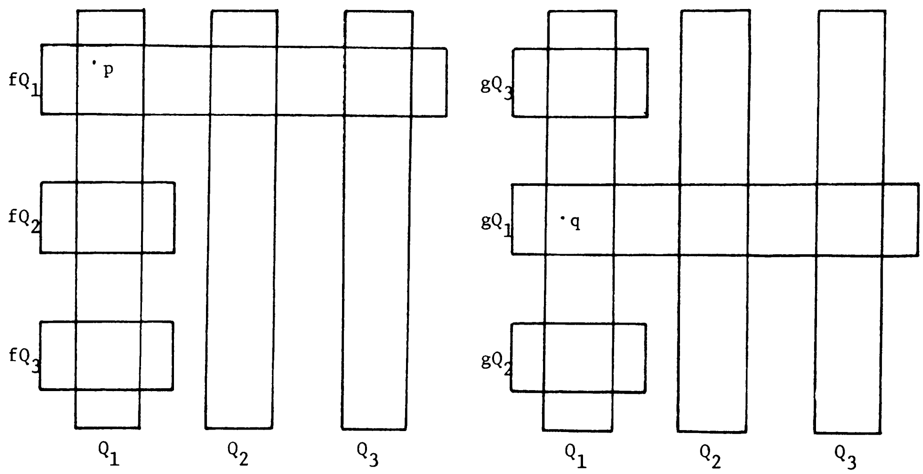


FIGURE 5.1.

To handle this situation, suppose W^+ is one dimensional. Then each $Q^+(x)$ is an oriented arc, so it has a linear order. We redefine A^+_{jk} as 0 or $\pm r$, where the sign is determined as before and r is the ordinal position of $fQ_j \cap Q^+(fx)$ in $Q^+(fx)$ for $x \in Q_j \cap f^{-1}Q_k$. If W^- is one dimensional we redefine A^- similarly, using the ordinal position of $Q^-(x) \cap f^{-1}Q_k$ in $Q^-(x)$.

(5.6) THEOREM. *Suppose f, Q, $A^\pm$ are as above, and Q_g, $A^\pm_g$ are corresponding data for a basic set Λ_g of some diffeomorphism g. Suppose, moreover, that the $Q^\pm(x)$ and $Q^\pm_g(x)$ are smooth disks. If $A^\pm = A^\pm_g$ and the dimensions of the splittings agree then there is a homeomorphism h: $Q_g \to Q$ with $hQ^\pm_g(x) = Q^\pm(hx)$ and $hg = fh$.*

PROOF. This comes from (4.6). The requisite $h^\pm_0$ are easily constructed from the information in $A^\pm$ using uniqueness of disks. □

In low dimensions we expect tameness. Thus we have

(5.7) THEOREM. *A zero-dimensional basic set Λ is dynamically tame in W^+ if Λ is finite or* $\dim W^+_\alpha \leqslant 2$.

PROOF. If Λ is finite use $Z^+ = \mathrm{Cl}\, W^+_\beta$ for $\beta < \alpha$. If $\dim W^+_\alpha = 0$ then Λ is finite, and if $\dim W^+_\alpha = 1$ then, for any Markov neighborhood Q, each $Q^+(x)$ is an arc.

So assume $\dim W^+_\alpha = 2$ and let Q be an approximately smooth Markov neighborhood. Let $R = \bigcap\{f^kQ\colon 0 \leqslant k \leqslant p\}$, and choose p large enough that each $R^+(x)$ lies in a smooth disk in $Q^+(x)$. Let Q_k be a component of Q, take $x \in Q_k \cap \Lambda$, and label the components of $R \cap Q_k(x)$ as R_{kj}, $1 \leqslant j \leqslant m$. Then $R \cap Q_k = \bigcup[Q^-(x), R_{kj}]$ and each R_{kj} is a disk with holes. Obtain disks D_{kj} from each R_{kj} by filling in the holes, and set $S = \bigcup_{kj}[Q^-(x), D_{kj}]$. Now it may happen that D_{kj} and D_{ki} meet, but in that case, by considering boundaries, it is clear that one must lie interior to the other—that is, R_{kj} was in one of the holes of R_{ki}, or vice-versa. Hence S is rectangular, each $S^+(x)$ is a smooth disk, and, again considering boundaries, we check $fS^+ \leqslant \mathrm{int}\, S^+$, so S is Markov. □

Question. Dynamic tameness is the important concept in applications, so it would be satisfying to reduce m to 1 in (5.3). Is this possible? A related question is: If f: $\mathbf{R}^n \to \mathbf{R}^n$ is a diffeomorphism and $\lim f^kD^n = 0$ then is there a smooth n-disk Δ with $f\Delta \subset \mathrm{int}\, \Delta$? This is true for $n \leqslant 3$, but I do not know about higher dimensions.

6. Tameness and transversality. The main result of this section shows how to push stable and unstable laminations of basic sets into general position. For a basic set Λ we write $W^\pm_\Lambda = \bigcup\{f^mW^\pm_{\Lambda\alpha}\colon m \in \mathbf{Z}\}$. These need not be laminations, but they are immersed submanifolds.

(6.1) THEOREM. *Let Λ, Δ be zero-dimensional basic sets with Λ tame in W^+_Λ, Δ tame in W^-_Δ, and $W^-_{\Lambda\alpha}$ a smooth lamination. Suppose K, C are compact, $V_0 \supset K$, $U_0 \supset C$ are open, $W^-_\Lambda \pitchfork W^+_\Delta$ in V_0, and, for some $\alpha > 0$, $m \in \mathbf{Z}^+$,*

(a) $W^-_\Lambda \cap \bar{U}_0 \subset f^mW^-_{\Lambda,\alpha} \backslash W^-_{\Lambda,\alpha}$,

(b) $W^+_\Delta \cap \bar{U}_0 \subset f^{-m}W^+_{\Delta,\alpha} \backslash W^+_{\Delta,\alpha}$.

Then for any $\varepsilon > 0$ there is an ε-push h_t with support in $U_0 \backslash K$ such that $h_1W^-_\Lambda \pitchfork W^+_\Delta$ in a neighborhood of $K \cup C$.

REMARK. Examples (8.2), (8.3) show that the tameness assumptions are necessary and that h_t cannot generally be C^1 small.

PROOF. Replacing f by some f^p we may simplify the hypotheses, replacing m by 1 and, using (5.3), "tame" by "dynamically tame". For p large enough we also have (3.1) for $W^-_{\Lambda\alpha}$ and $W^+_{\Delta\alpha}$.

We first prove the following claim: if $\varepsilon > 0$ and N is a submanifold with $N \cap \overline{U_0}$ compact and $W^-_\Lambda \pitchfork N$ in V_0, then there is an ε-push H_t with support in $U_0 \setminus K$ such that $W^-_\Lambda \pitchfork H_1 N$ in a neighborhood of $K \cup C$. In this proof we will not use the smoothness assumption on $W^-_{\Lambda\alpha}$.

The first step is the construction of an appropriate coordinate system. We first apply (3.5), using (a), to find smaller open neighborhoods U_1 of C, V_1 of K and a semi-invariant tubular family $\mathfrak{L}^+_\Lambda$ for $fW^-_{\Lambda\alpha}$ with $\mathfrak{L}^+_\Lambda \leqslant N \cap V_1$ so that $\mathfrak{L}^+_\Lambda \cap U_1$ is smooth. Take an approximately smooth Markov neighborhood Q_0 for Λ with $Q_0^+ \leqslant \mathfrak{L}^+_\Lambda$ and each $Q_0^+(x)$ a smooth disk. Setting $Q_1 = f^k Q$ we have $\mathrm{Cl}\, fW^-_{\Lambda\alpha} \subset \mathrm{int}\, Q_1$ for k large enough.

Now select open neighborhoods U_j of C, V_j of K for $2 \leqslant j \leqslant 6$ with $U_j \supset \mathrm{Cl}\, U_{j+1}$, $V_j \supset \mathrm{Cl}\, V_{j+1}$ for $1 \leqslant j \leqslant 5$. Set $Q = \bigcap \{f^i Q_1 \colon 0 \leqslant i \leqslant r\}$, $\delta = \max \operatorname{diam} Q^+(x)$. Then $\delta \to 0$ as $r \to \infty$ so we may assume $\delta < \varepsilon/3$ and

(c) $\delta < \frac{1}{2} d(U_{j+1}, \partial U_j)$, $\delta < \frac{1}{2} d(V_{j+1}, \partial V_j)$, $1 \leqslant j \leqslant 5$.

Next select $\xi_i \in \Lambda$, one in each component of Q, and set $P_j = [\bigcup Q^-(\xi_i)] \cap U_j$. We regard P_1 as an open subset of $\mathbf{R}^u$ and fix diffeomorphisms $\phi_i \colon D^s \to Q^+(\xi_i)$. Using a smooth approximation $\tilde{Q}$ to Q we have a diffeomorphism $\tilde{\phi} \colon P_1 \times D^s \to M$ defined by $\tilde{\phi}(x, y) = [x, \phi_i y]^{\sim}$ for $x \in Q^-(\xi_i) \cap P_1$. We may assume $\tilde{Q}^+(x) = Q^+(x)$ off a neighborhood of W^+_Λ and $\tilde{Q}^-(x) = Q^-(x)$ off fQ, so $\tilde{\phi}(x \times D^s) = Q^+(x)$ for $x \in P_1$.

With respect to $\tilde{\phi}$, P_1 is given as a graph: $P_1 = \{\tilde{\phi}(x, wx) \colon x \in P_1\}$ where $w \colon P_1 \to \mathrm{int}\, D^s$ is smooth. Select a smooth $\alpha \colon D^s \times \mathrm{int}\, D^s \to D^s$ so that $y \mapsto \alpha(y, z)$ is a diffeomorphism of D^s onto D^s with $\alpha(0, z) = z$. Setting $\phi(x, y) = \tilde{\phi}(x, \alpha(y, w(x)))$ we have a diffeomorphism with $\phi(x, 0) = x$ and $\phi(0 \times D^s) = Q^+(x)$. If $u \in U_3 \cap W^-_\Lambda$ then $u \in Q^+(x)$, $x \in P_2$ by (c) and the definition of δ. Hence $U_3 \cap W^-_\Lambda \subset \mathrm{int}\, \mathrm{Im}\, \phi$, so we need only push N transerve to W^-_Λ inside $\mathrm{Im}\, \phi$.

We construct H in three stages. First, by standard transversality theory there is an $\varepsilon/3$ push H^1_t with support in $U_1 \setminus V_1$ so that $H^1_1 N$ is transverse to P_2. We still have $Q^+(x) \subset H^1_1 N$ for $x \in V_2 \cap P_2$ so we may apply (2.2), (2.4) to obtain a smooth tubular family $\mathfrak{F}$ for P_2 with $\mathfrak{F} \leqslant \mathrm{int}\, Q^+ \cap V_3$, $\mathfrak{F}(x) \subset H^1_1 N$ for $x \in P_3$, and $\operatorname{diam} \mathfrak{F}(x) < \delta$ for all x. Then there is $r > 0$ such that $\phi(P_4 \times rD^s) \subset \mathfrak{F}$ and $\phi(P_4 \times y) \pitchfork \mathfrak{F}$ for $y \in rD^s$. Hence we can define $\beta \colon P_5 \times rD^s \to \mathbf{R}^u$ by

$$\mathfrak{F}(x) \cap \phi(P_4 \times y) = (x + \beta(x, y), y).$$

We have $\beta(x, y) = 0$ for $x \in P_5 \cap V_3$ since $\mathfrak{F}(x) \subset Q^+(x)$ for such x, and $\beta(x, 0) = 0$. Take $\lambda \colon P_1 \to [0, 1]$, $\mu \colon R^s \to [0, 1]$ with $\lambda = 1$ on P_6, $\lambda = 0$ off P_5, $\mu = 1$ on $\frac{1}{2} rD^s$, $\mu = 0$ off rD^s, and set

$$H^2_t \phi(x, y) = \phi(x + t\lambda(x)\mu(y)\beta(x, y), y).$$

Since H_t^2 has support in $(\operatorname{int}\operatorname{Im}\phi)\backslash V_4$ it extends to all of M. To see it is an isotopy it is sufficient to check that, for r small enough, each $x \to H_t^2(x, y)$ is an embedding. But this follows from openness of embeddings, since λ is fixed and $x \mapsto \beta(x, y)$ is C^1 close to 0 for r small. Clearly, for r small enough, H_t^2 is an $\varepsilon/3$ push with support in $U_4\backslash V_4$, and it has the following effect:

(d) if $x \in P_6$ and $\phi(x \times \frac{1}{2}rD^s)$ meets $H_1^2H_1^1N$ then $\phi(x \times \frac{1}{2}rD^s) \subset H_1^2H_1^1N$. The third $\varepsilon/3$-push H_t^3 is left to the reader: it should preserve the Q^+ factors and satisfy $H_1^3\phi(x \times \frac{1}{2}rD^s) \supset \phi(x \times RD^2)$ for $x \in P_6\backslash V_6$, where $R < 1$ is large enough that W_Λ^- misses $\phi(P_6 \times (D^s\backslash RD^s))$. We let H_t be the ε-push defined by concatenating H_t^1, H_t^2, H_t^3. If H_1N meets $W_\Lambda^- \cap \phi(P_6 \times RD^s)$ in $\phi(x, y)$ then either $x \in V_6$, in which case $H_1N \supset Q^+(x)$, or $H_1^2H_1^1N$ meets $\phi(x \times \frac{1}{2}rD^s)$ so, using (d), $H_1N \supset \phi(x \times RD^s)$. In either case we have shown H_1N is transverse to W_Λ^- at $\phi(x, y)$, and we have established the claim.

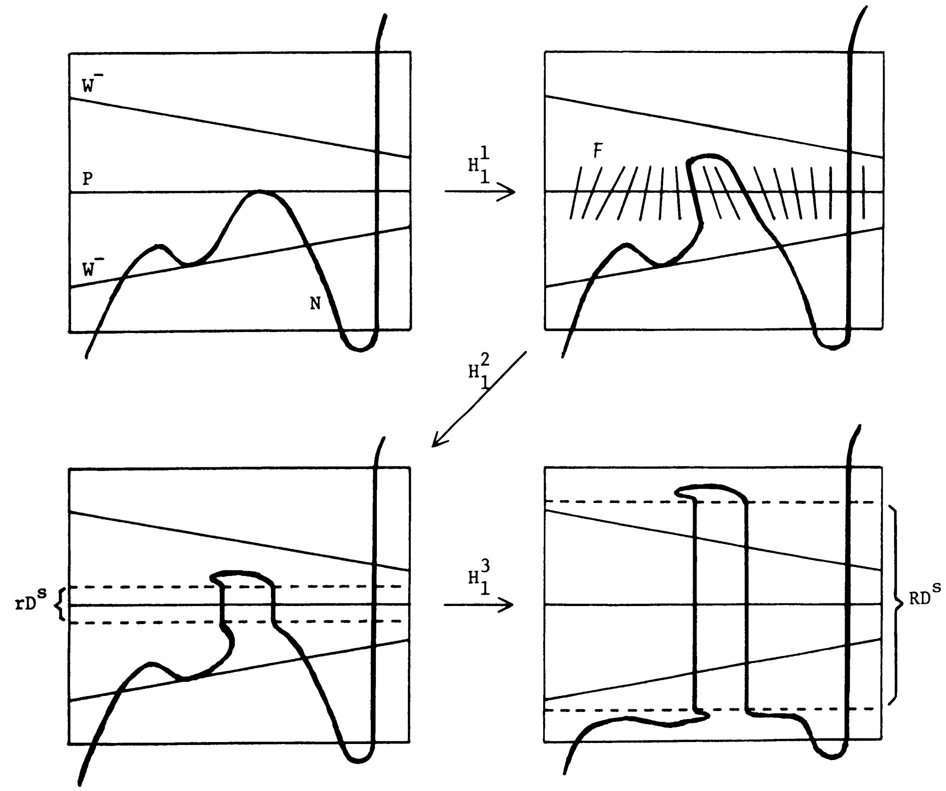


FIGURE 6.1.
The construction of H (Theorem 6.1) using ϕ coordinates. The vertical factor is D^s.

To prove the theorem we repeat the above procedure, replacing W_Λ^- by W_Δ^+ and N by W_Λ^-. We find $\mathfrak{L}_\Delta^- \leq fW_{\Lambda,\alpha}^- \cap V_1$ by (3.5), using now the smoothness assumption, and, from (6.1)(a), we have compactness of $fW_{\Lambda,\alpha}^- \cap \overline{U}_0$. Then construct Q and P_j as above, interchanging $+$ and $-$. We construct h_t in three stages as above. However, we cannot use ordinary transversality to construct h_t^1 so that $h_1^1 W_\Lambda^- \pitchfork P_2$. Instead we use the claim, which provides an $\varepsilon/3$ push H_t with $W_\Lambda^- \pitchfork H_1 P_2$, and set $h_t^1 = H_t^{-1}$. The construction of h_t^2, h_t^3 is now exactly similar to the above, where again we require the smoothness assumption in constructing $\mathfrak{F}$. $\square$

Question. The tameness assumptions in (6.1) are assumptions on the laminations $W_{\Lambda,\alpha}^-$. For example, we could define a lamination $\mathfrak{L}$ to be transversely tame iff $\mathfrak{L}$ meets every transversal submanifold T of complementary dimension in a zero-dimensional set which is tame in T. Is there a suitable transversality theory for pairs of such laminations? If one lamination is zero dimensional (5.1)(c) can be expanded to give a satisfactory answer.

The smoothness assumption in (6.1) is inconvenient, but I do not know how to prove the theorem without it, so the following will be necessary in the next section.

(6.2) THEOREM. *Suppose Λ_0 is a zero-dimensional basic set for f_0 which is dynamically tame in both W^+ and W^-; suppose $\varepsilon > 0$ and V is an isolating neighborhood of Λ_0. Then there is an ε-isotopy f_t starting at f_0 with support in V such that f_1 is C^2 and satisfies both strong contraction conditions* (3.8) *on $\Lambda_1 = I(V, f_1)$. Moreover, f_0 and f_1 are topologically conjugate on neighborhoods of Λ_0 and Λ_1, and Λ_1 is dynamically tame in both W^+ and W^-.*

REMARKS. The strong contraction conditions are the only known general conditions assuring smoothness of $W_\alpha^\pm$. They clearly require C^1 large moves. Wildness can interfere with the contraction conditions, as in [**RW**].

PROOF. By a C^1 small preliminary isotopy we adjust f_0 to be C^2. This does not change local conjugacy type or tameness, by (4.7), (5.5).

Take an approximately C^2 Markov neighborhood $Q \subset \operatorname{int} V$ with disk factors, with components Q_j so that $\operatorname{diam} Q_j < \varepsilon/2, \operatorname{diam} f_0 Q_j < \varepsilon/2$. Let $\tilde{Q}$ be a C^2 approximation with $\tilde{Q}^\pm(x) = Q^\pm(x)$ for $x \in Q \setminus f_0^{\mp 3} Q$ and $f_0^{\pm 1}\tilde{Q}^\pm \pitchfork \tilde{Q}^\mp$. Using canonical coordinates from $\tilde{Q}$ we identify Q with $\bigcup (e_j + D^u) \times D^s \subset \mathbf{R}^{u+s}$, where $e_j = (3j, 0, \ldots, 0) \in \mathbf{R}^u$. These are C^2 coordinates and, writing $f_0(x, y) = (f^-(x, y), f^+(x, y))$, we have

$$Df_0 = \begin{bmatrix} D_1 f^- & D_2 f^- \\ D_1 f^+ & D_2 f^+ \end{bmatrix},$$

and the assumptions $f^{\pm 1}\tilde{Q}^\pm \pitchfork \tilde{Q}^\mp$ are equivalent to invertibility of $D_1 f^-$, $D_2 f^+$.

Now we add extra contraction and expansion: Pick $\alpha > 1$, $r < 1$. Let Ψ_t^+ be an isotopy of D^s which is fixed near ∂D^s with $\Psi_0^+ = 1$ and $\Psi_1^+(y) = \alpha^{-1} y$ for $y \in rD^s$. Similarly take Ψ_t^- on D^u with $\Psi_1^-(x) = \alpha x$ for $x \in \alpha^{-1} r D^u$. Pick $\lambda^+ : D^s \to [0, 1]$ to be 0 near ∂D^s and 1 on rD^s, and similarly $\lambda^-: D^u \to [0, 1]$. Define $\phi_t^\pm$ on Q by

$$\phi_t^+(e_j + x, y) = \big(e_j + x, \Psi^+(y, t\lambda^-(x))\big),$$
$$\phi_t^-(e_j + x, y) = \big(e_j + \Psi^-(x, t\lambda^+(y)), y\big).$$

These extend by the identity to M, and we can define our isotopy by $f_t = \phi_t^- f_0 \phi_t^+$.

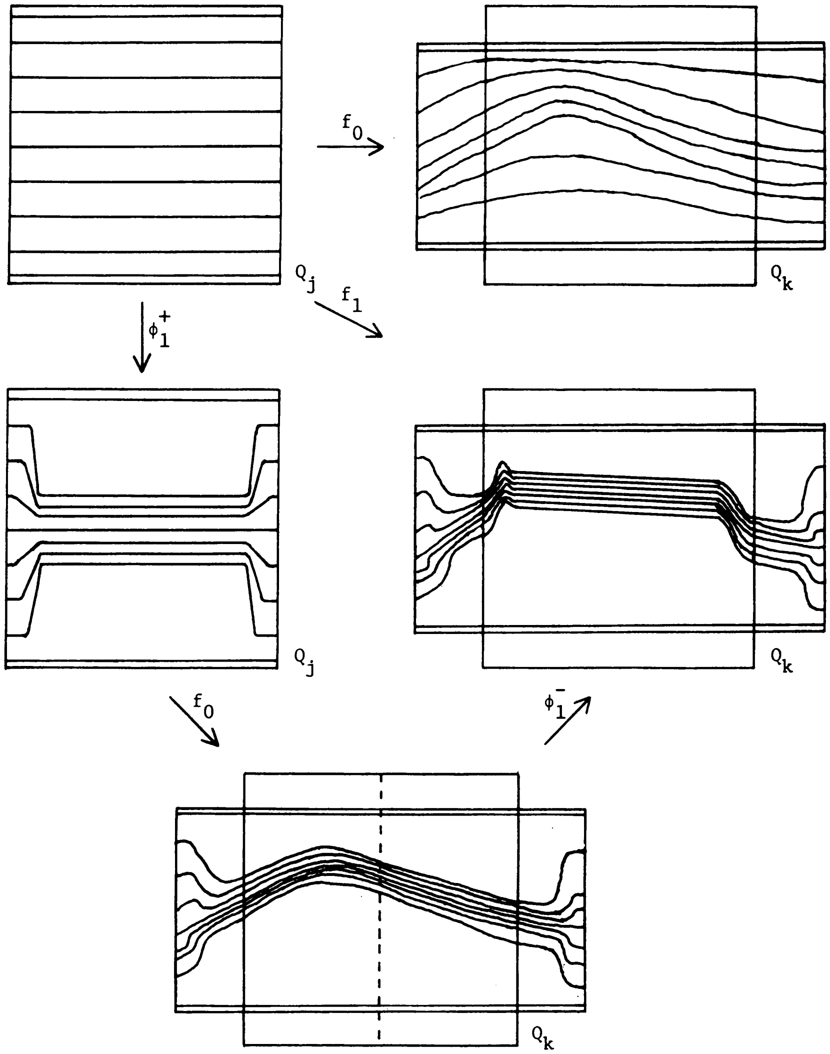


FIGURE 6.2.
Constructing f_1 in Theorem 6.2. Shown are the effects on the horizontal lamination $\tilde{Q}^-$.
Note the lack of control near the edges.

Since the $\phi^\pm$ preserve the components of Q this satisfies $d(f_t x, f_0 x) < \operatorname{diam} Q_k + \operatorname{diam} f_0 Q_j < \varepsilon$ for $x \in Q_j \cap f_0^{-1} Q_k$, and

$$d(f_t x, f_0 x) < \operatorname{diam} f_0 Q_j < \varepsilon \quad \text{for } x \in Q_j \setminus f_0^{-1} Q.$$

For r close enough to 1 the set $Q_r = \bigcup (e_j + rD^u) \times rD^s$ is still an approximately C^2 Markov neighborhood of Λ_0 and for $(x, y) \in Q_r \cap f_1^{-1} Q_r$ we have

$$Df_1(x, y) = \begin{bmatrix} \alpha D_1 f^-(x, \alpha^{-1} y) & D_2 f^-(x, \alpha^{-1} y) \\ D_1 f^+(x, \alpha^{-1} y) & \alpha^{-1} D_2 f^+(x, \alpha^{-1} y) \end{bmatrix}.$$

Straightforward calculations based on [**HP**, §4], using invertibility of $D_1 f^-$, $D_2 f^+$ and compactness, show that, for α large, $\Lambda_1 = I(Q_r, f_1)$ is hyperbolic. Moreover, the strong contraction conditions (3.8) are satisfied and the splitting $E_1^- \oplus E_1^+$ is close to $\mathbf{R}^u \oplus \mathbf{R}^s$.

Because $f_1 = f_0$ off $Q \cup f_0^{-1}Q$ we have $I(V, f_1) = I(Q, f_1)$. For r close enough to 1 and $x \in Q \backslash Q_r$ we have $f_1 x \notin Q$ or $f_1^{-1}x \notin Q$, and hence $I(Q_r, f_1) = I(Q, f_1)$. Thus we will finish (6.2) by showing $f_0 \mid Q_r$ and $f_1 \mid Q_r$ are conjugate. For this we only need to make Q_r into a Markov neighborhood for f_1 and apply (5.6)—it is easy to check that the algebraic data for f_0 and f_1 are identical. Set $R = \bigcup (e_j + rD^u) \times D^s$ and $R_0^-(e_j + x, y) = (e_j + rD^u) \times y = \tilde{Q}^-(e_j + x, y) \cap R$. Then for $z = (e_j + x, y)$ with y close to ∂D^s we have $\phi_1^+ z = z$ so

$$f_1 R_0^-(z) \supset \phi_1^- f_0 R_0^-(z) \supset \phi_1^- R_0^-(f_0 z) \supset R_0^-(f_0 z).$$

Hence $R \backslash f_1 R = R \backslash f_0 R$ and we can apply the graph transform methods of (3.4) to define disks R_k^- converging to R^- with $f_1 R^-(z) \supset R^-(f_1 z)$. Moreover, the same estimates that assured strong contraction for f_1 show that the disks R^- are approximately horizontal: that is, they are given as graphs of maps $rD^u \to D^s$ with small derivatives. We similarly obtain disks R^+ which are approximately vertical, so $R^+ \pitchfork R^-$. We have $R^{\pm}(z) \subset Q_r$ if $z \in Q_r$, so these disks provide the required Markov neighborhood structure for f_1. $\square$

7. Some global perturbations. We now apply the local result of §6 to some global questions. In a sense we want a (very weak) Kupka-Smale theorem for zero-dimensional basic sets.

We recall, from [**B**] or [**S**], some basic results. The nonwandering set $\Omega(f)$ is $\{x \in M\colon$ for any neighborhood U of x there is $k > 0$ with $f^k U \cap U \neq \varnothing\}$. A finite sequence $\{M_j\}$ of compact submanifolds of M is a filtration for f iff $fM_j \subset \operatorname{int} M_j$, $M_j \subset \operatorname{int} M_{j+1}$ for all j, and

$$\Omega_j = \Omega \cap M_j \backslash M_{j-1} = I(M_j \backslash M_{j-1}, f).$$

An alternative to Smale's original definition of an AN (Axiom A, no cycle) diffeomorphism is that there exists a filtration with each Ω_j a basic set. Similarly, an AS (Axiom A, strong transversality) diffeomorphism is one with Ω a disjoint union of basic sets whose invariant laminations are transverse. We say f is structurally stable (Ω-stable) iff for all g in some neighborhood of f, f and g (respectively $f \mid \Omega(f)$ and $g \mid \Omega(g)$) are topologically conjugate. Then $AS \Rightarrow AN \Rightarrow \Omega$-stability (Smale [**S, S1**]), and $AS \Rightarrow$ structural stability (Robbin [**Ro**] and Robinson [**R**]).

If f is AN and $\dim \Omega = 0$ we say Ω is (dynamically) tame in $W^{\pm}$ iff each Ω_i is, and we say Ω has smooth invariant laminations iff each Ω_i does.

(7.1) THEOREM. *Suppose f_0 is AN with* $\dim \Omega(f) = 0$. *Suppose $\varepsilon > 0$ and V is a neighborhood of* $\Omega(f_0)$.

(a) *If $\Omega(f_0)$ is dynamically tame in W^+ and W^- then there is an ε-isotopy f_t starting at f_0 with support in V, such that f_0 and f_1 are topologically conjugate on neighborhoods of $\Omega(f_0)$ and $\Omega(f_1)$, and $\Omega(f_1)$ is dynamically tame in W^+ and W^- and has smooth invariant laminations.*

(b) *If $\Omega(f_0)$ has smooth invariant laminations and is tame in W^+ and W^- then there is an ε-isotopy f_t starting at f_0 with support in $V \setminus U_0$ such that each f_t is AN and f_1 is AS.*

REMARKS. Without tameness (b) is false, although it may hold if control on the isotopy is relaxed; see (8.3). If $\dim \Omega > 0$ the situation is hopeless: in **[FR]** there is an AN diffeomorphism which is not Ω-conjugate to any AS diffeomorphism (on the same manifold).

The smoothness assumption is always satisfied if $\dim M \leqslant 2$ and, by (5.7), the tameness assumptions here and in (7.2) are satisfied if $\dim M \leqslant 3$.

PROOF. Part (a) is immediate from (6.2). To reduce (b) to (6.1) we need a standard argument (Palis induction). Suppose F_k^- is a compact proper fundamental domain for W_k^- with $f^2F_k^- \cap F_k^- = \varnothing$. Suppose $W_k^- \cap W_j^+$ for $k < j < l$. Then $K_0 = F_k^- \cap \bigcup \{W_j^+ : k < j < l\}$ is compact and, if V is a sufficiently small neighborhood of K_0, we have $(W_k^- \cap V) \pitchfork W_l^+$ and $(W_l^+ \cap F_k^-) \setminus V \subset f^{-m} W_{l,\alpha}^+$ for some m. For all this see **[R**, §4]. Now we can apply (6.1) twice, first with $K_1 \subset V$ a compact neighborhood of K_0 and $C_1 = F_k^- \cap fF_k^- \setminus \operatorname{int} K_1$, and then with $C = F_k^- \setminus$ a neighborhood of $C_1 \cup fC_1$, and $K = K_1 \cup C_1$. Realizing the isotopies of (6.1) as isotopies of f is simple. After this we have $W_k^- \pitchfork W_l^+$, and since we did not modify f on K_0 we still have $W_k^- \pitchfork W_j^+$ for $k > j > l$. Moreover, considering the filtration, we see that W_m^- is not modified for any $m < k$; neither is $W_m^+ \cap M_{k-1}$ for $m < k$. Thus we can prove part (b) by a double induction: for fixed k proceed downward in j until $W_k^- \pitchfork W_j^+$ for all $j < k$. Then proceed upwards in k until Ω is exhausted. □

Our other application is a relativization of the Smale-Shub-Sullivan-Williams-Zeeman isotopy theory (see **[S2, SS, Z]**). Say $f_0, f_1 \in \operatorname{Diff} M$ are isotopic rel g, where $g: U \to M \supset U$, if there is an isotopy f_t between them with $f_t | U = g$ for all t.

(7.2) THEOREM. *Suppose U is closed in M, $gU \subset \operatorname{int} U$, and J is an equivalence class under isotopy* rel g.

(a) *If g is Ω-stable then the Ω-stable diffeomorphisms are C^0 dense in J.*

(b) *If g is AS, $\dim \Omega = 0$, and Ω is tame in W^- then the AS diffeomorphisms are C^0 dense in J.*

In particular, such g extends to an Ω-stable or AS diffeomorphism, as the case may be, if it extends to any diffeomorphism.

PROOF. Part (a) is a trivial modification of Smale's original construction. The point is that, in isotoping an extension f_0 of g to get Ω-stability, $f_0 | U$ need never be disturbed. Then $f_1 | M \setminus U$ is Ω-stable by construction while $f_1 | U$ is Ω-stable by hypothesis, and U can be taken as a level of the filtration. To get C^0 density just use small handles as in [SS].

For part (b) we isotope f_0 as in part (a) to produce Ω-stability for f_1. An examination of Smale's procedure shows that $\Omega \setminus U$ is dynamically tame in $W^{\pm}$. Applying (7.1)(a) we get strong contraction without disturbing $f_1 | U$. Finally we repeat the induction in (7.1)(b), noticing that it has been initialized, by hypothesis, for basic sets in U. This process takes place near the basic sets in $M \setminus U$ and uses the full force of (6.1): that is, smoothness is only required for one basic set when a connection between two is made transversal. □

Question. If g is Morse-Smale (i.e., AS plus finite Ω) then does g have a Morse-Smale extension?

8. Examples.

(8.1) A nonextendible lamination.

We first construct a map ϕ: $K \times \mathbf{R} \to \mathbf{R}^3$ with each ϕ_t an embedding and $d\phi_t/dt$ continuous on $K \times \mathbf{R}$, so that $\phi_t K$ is a compact zero-dimensional set which is tame for $t < 1$ and wild for $t \geqslant 1$.

In this discussion torus means solid torus of revolution. Start with a torus T^0 and inside it arrange tori T_k^1 as in Figure 8.1(a); they are alternately horizontal and meridional. We start with $\phi_0 = 1$ and define ϕ_t, $0 \leqslant t \leqslant \frac{1}{2}$, by rotating the meridional tori through 90° to obtain the chain in Figure 8.1(b). This of course is not an isotopy on $\cup T_k^1$. However, if the next generation of tori T_j^2 (which lie analogously in the T_k^1) are small enough, thin enough, and carefully arranged, then they will miss each other during the rotations of the T_k^1. Hence ϕ_t defines an isotopy of $\cup T_j^2$. Next we define ϕ_t for $\frac{1}{2} \leqslant t \leqslant \frac{3}{4}$ to link up the T_j^2, and then place the T_i^3 so that $\phi_t | \cup T_i^3$ is an isotopy. Continuing this process defines $K = \cap_n(\cup_i T_i^n)$ and ϕ_t for $0 \leqslant t < 1$. The reader may check that $\phi_1 = \lim_{t\to 1} \phi_t$ is an embedding and that $\lim_{t \to 1} d\phi_t/dt = 0$ uniformly if the sizes of the T_i^n approach 0 rapidly enough. We set $\phi_t = \phi_1$ for $t > 1$, $\phi_t = \phi_0$ for $t < 0$.

Now define a lamination $\mathfrak{L}$ in $\mathbf{R}^4$ by $\mathfrak{L} = \{(\phi_t x, t)\colon (x, t) \in K \times \mathbf{R}\}$, and set $N_t = \mathbf{R}^3 \times t$. We note the following properties:

(a) $\mathfrak{L}$ does not extend to a lamination of an open set: for then the holonomy map would provide a homeomorphism from a neighborhood of K in N_0 to a neighborhood of $\phi_1 K$ in N_1, taking K to $\phi_1 K$. This is impossible.

(b) Although $\mathfrak{L}$ is smooth for $t < \frac{1}{2}$ and $t > \frac{3}{2}$ there is no smooth lamination homeomorphic to $\mathfrak{L}$ and agreeing with $\mathfrak{L}$ for $t \leqslant 0$ and $t \geqslant 2$, since smooth laminations extend to open sets.

(c) There is no tubular family $\mathfrak{L}_1$ for N_1 with $\mathfrak{L}_1 \leqslant \mathfrak{L}$, so (2.2) does not hold. In fact,

(d) If X is a plaquation of $\mathfrak{L}$ restricted to $\mathfrak{L} \cap N_1$ then X is a disjoint disk family but no extension of X to an open set in N_1 is disjoint.

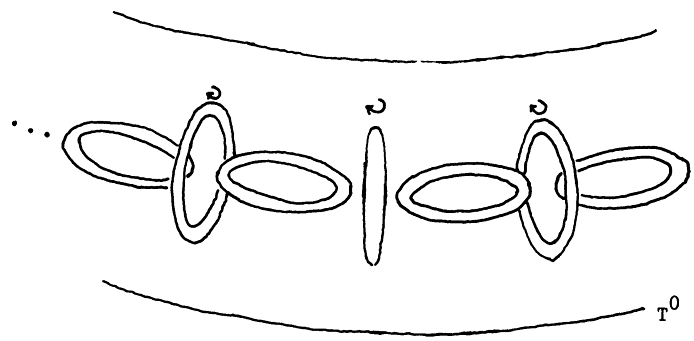
Question. Suppose $\mathfrak{L}$ is a lamination which extends to an open set. Can $\mathfrak{L}$ then be smoothed as in (b)? Does (2.2) hold for such laminations?

(8.2) A C^1-thick basic set.

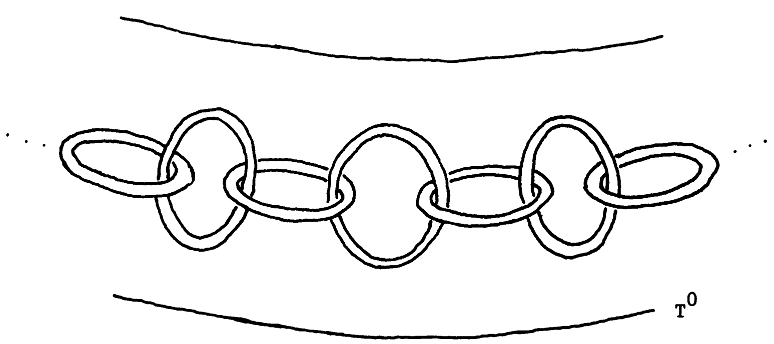
Let $B_0 = [0, 1]^2 \subset R^2$ and set $B_{1k} = [a_k, a_k + \alpha] \times [b_k, b_k + \beta]$ for $k = 1, 2$, where

$$0 < a_1 < a_1 + \alpha < a_2 < a_2 + \alpha < 1, \qquad \beta \geqslant 3/4,$$
$$0 < b_1 < 1/3(1 - \beta) < 2/3(1 - \beta) < b_2 < 1 - \beta < 1.$$

See Figure 8.2. By the middle third of a rectangle $J \times [r, r + s]$ we mean $J \times [r + s/3, r + 2s/3]$. The inequalities above have been chosen so that the reader may verify: If a straight line of slope λ, $|\lambda| \leqslant (1 - \beta)/12$, meets the middle third of B_0



(a) t = 0.



(b) t = 1/2.

FIGURE 8.1.
Closing the chain.

then it meets the middle third of either B_{11} or B_{12}. Now let ϕ: $B_{11} \cup B_{12} \to B_0$ be an affine homeomorphism on each component preserving the horizontal and vertical foliations. By induction it follows that a straight line with slope λ, $|\lambda| \leqslant (1-\beta)\beta^m/(12\alpha^m)$, which meets the middle third of some component B_{mk} of $B_m = \phi^{-m}B_0$ must meet the middle third of some component of $B_{mk} \cap B_{m+1}$. Hence, with $\Lambda_0 = \bigcap\{\phi^{-j}B_0 \colon j \geqslant 0\}$, we have (from the Mean Value Theorem):

(*) Given $L > 0$ there is $\varepsilon > 0$ such that, if $\gamma(t) = (t, \sigma(t))$ and $|\sigma(t) - \frac{1}{2}| < \varepsilon$, $|\sigma'(t)| < L$ for all t, then $\operatorname{Im}\gamma \cap \Lambda_0 \neq \varnothing$.

We note that Λ_0 has zero Lebesgue measure and is tame. We obtain a diffeomorphism by defining $f(x, y, z) = (\phi(x, y), (-1)^k(1+z)/4)$ on $B_{1k} \times [-1, 1]$ and applying the isotopy extension theorem. Then f has a basic set Λ with $W^+ = \Lambda_0 \times [-1, 1]$ locally. From (*) we readily deduce that any C^1 small perturbation of the curve $t \to (t, \frac{1}{2}, 0)$ meets the one-dimensional lamination W^+.

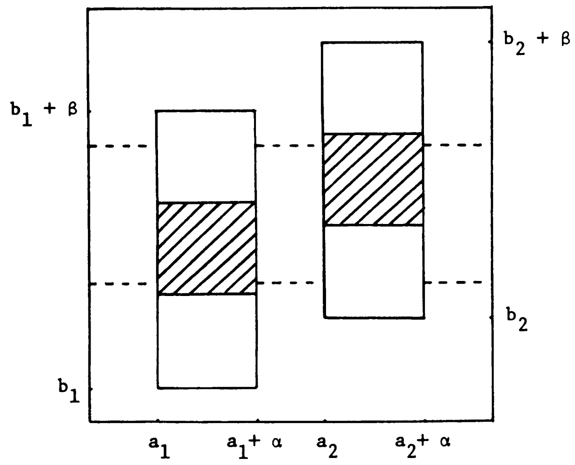


FIGURE 8.2.
B_0, B_{11}, B_{12} and their middle thirds. Here $\alpha = \frac{1}{4}, \beta = \frac{3}{4}$.

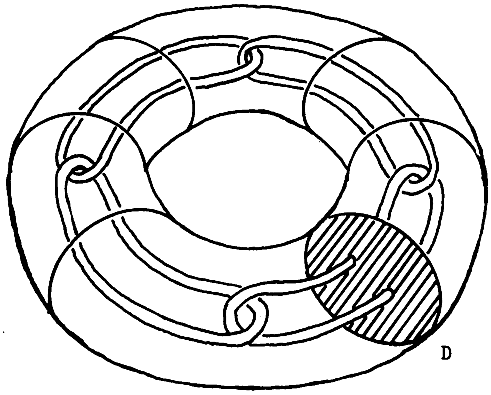


FIGURE 8.3.
The large solid torus is $Q^+(x)$ and inside it are the four components of $Q^+(x) \cap fQ$.

Another curiosity: The strong unstable set W^{--} corresponding to the dominant eigenvalue α^{-1} of $Df|\Lambda$ is the union of a one-dimensional disk family through Λ, but is two dimensional as a subset of R^3.

Question. This example illustrates the necessity of C^1-large isotopies in (6.1), and hence in (7.1), (7.2), but the possibility remains that C^1-small isotopies which modify f on a neighborhood of Λ may suffice. The properties of our example persist under small C^2 perturbations by arguments as in [N1]. Is there an example which persists under C^1 perturbation?

(8.3) Wild connections.

Newhouse [N] describes a basic set Λ for a diffeomorphism of S^4 with a Markov neighborhood Q displaying the structure in Figure 8.3. We can easily introduce a fixed point p with two-dimensional stable manifold so that, for some $x \in Q \setminus f^{-1}Q$, W_p^+ contains the disk D shown in Figure 8.3. Pick $\xi \in \Lambda$ in the component Q_0 of Q containing x and let $\pi\colon Q_0 \to Q^+(\xi)$ be the projection. Then for any C^0 small perturbation $\tilde{f}$ of f which is fixed on a neighborhood of $Q \cup p$, $\tilde{W}_p^+$ contains a disk $\tilde{D}$ in Q_0 near D, so $\pi\partial\tilde{D}$ is near $\pi\partial D$ in $Q^+(\xi)$. But $\pi\partial D$ is not contractible in $Q^+(\xi)\setminus\Lambda$ so $\pi\tilde{D}$ meets Λ. Hence $\tilde{W}_p^+$ meets $\tilde{W}_\Lambda^-$, which is one dimensional, so $\tilde{W}_p^+$ is not transverse to $\tilde{W}_\Lambda^-$. This shows the necessity of tameness in §§6, 7.

Question. This obviously yields an example of structural instability. Is there an AN diffeomorphism with $\dim \Omega = 0$ which cannot be isotoped, within AN diffeomorphisms, to a structurally stable diffeomorphism?

References

[B] R. Bowen, *Equilibrium states and the ergodic theory of Anosov diffeomorphisms*, Chap. 3, Lecture Notes in Math., vol. 470, Springer-Verlag, Berlin, 1975.

[BF] R. Bowen and J. Franks, *Homology for zero-dimensional basic sets*, Ann. of Math. (2) **106** (1977), 73–92.

[D] J. Dugundji, *An extension of Tietze's theorem*, Pacific J. Math. **1** (1951), 353–367.

[FR] J. Franks and C. Robinson, *A quasi-Anosov diffeomorphism that is not Anosov*, Trans. Amer. Math. Soc. **223** (1976), 267–278.

[H] M. Hirsch, *Differential topology*, Graduate Texts in Math., no. 33, Springer-Verlag, New York, 1976.

[HP] M. Hirsch and C. Pugh, *Stable manifolds and hyperbolic sets*, Proc. Sympos. Pure Math., vol. 14, Amer. Math. Soc., Providence, R.I., 1970, pp. 133–164.

[HPS] M. Hirsch, C. Pugh and M. Shub, *Invariant manifolds*, Lecture Notes in Math., vol. 583, Springer-Verlag, Berlin, 1977.

[HPPS] M. Hirsch, J. Palis, C. Pugh and M. Shub, *Neighborhoods of hyperbolic sets*, Invent. Math. **9** (1970), 121–134.

[K] L. Keldys, *Topological embeddings in Euclidean space*, Proc. Steklov Inst. Math. **81** (1966).

[M] W. de Melo, *Structural stability of diffeomorphisms on two-manifolds*, Invent. Math. **21** (1973), 233–246.

[N] S. Newhouse, *On simple arcs between structurally stable flows*, Dynamical Systems—Warwick 1974, Lecture Notes in Math., vol. 468, Springer-Verlag, Berlin, 1975, pp. 209–233.

[N1] ______, *Nondensity of Axiom $A(a)$ on S^2*, Proc. Sympos. Pure Math., vol. 14, Amer. Math. Soc., Providence, R. I., 1970, pp. 191–202.

[N2] ______, *The abundance of wild hyperbolic sets and non-smooth stable sets for diffeomorphisms*, Inst. Hautes Études Sci. Publ. Math. **50** (1979), 101–152.

[PS] J. Palis and S. Smale, *Structural stability theorems*, Proc. Sympos. Pure Math., vol. 14, Amer. Math. Soc., Providence, R. I., 1970, pp. 223–231.

[Ro] J. Robbin, *A structural stability theorem*, Ann. of Math. (2) **94** (1971), 447–493.

[R] C. Robinson, *Structural stability of C^1 diffeomorphisms*, J. Differential Equations **22** (1976), 28–73.

[RW] C. Robinson and R. F. Williams, *Classification of expanding attractors: an example*, Topology **15** (1976), 321–323.

[SS] M. Shub and D. Sullivan, *Homology theory and dynamical systems*, Topology **14** (1975), 109–132.

[S] S. Smale, *Differentiable dynamical systems*, Bull. Amer. Math. Soc. **73** (1967), 747–817.

[S1] ______, *The Ω-stability theorem*, Proc. Sympos. Pure Math., vol. 14, Amer. Math. Soc., Providence, R.I., 1970, pp. 289–298.

[S2] ______, *Stability and isotopy in discrete dynamical systems*, Symposium on Dynamical Systems—Salvador, Academic Press, New York, 1973, pp. 527–530.

[St] J. Stallings, *The piecewise-linear structure of Euclidean space*, Proc. Cambridge Philos. Soc. **58** (1962), 481–488.

[Z] E. C. Zeeman, *C^0 density of stable diffeomorphisms and flows*, Univ. of Warwick, preprint, 1973.

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